Curves in $\mathbb{R}^n$

1. Limits, Continuity and Differentiation.

Throughout this section, $I$ is an interval, $a \in I$ and

$$\mathbf{r} = (r_1, \ldots, r_n) : I \to \mathbb{R}^n.$$  

In 12.5 in the book $n = 3$. Often, for the sake of brevity, we will say $\mathbf{r}$ is a curve. Notice the difference between the range of $\mathbf{r}$ (which the book calls the track) and $\mathbf{r}$ itself.

Definition 1.1. Suppose $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{R}^n$. We say $\mathbf{r}(t)$ approaches $\mathbf{l}$ as $t$ approaches $a$ and write

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{l}$$

if for each $\epsilon > 0$ there is $\delta > 0$ such that

$$t \in I \text{ and } 0 < |t - a| < \delta \Rightarrow |\mathbf{r}(t) - \mathbf{l}| < \epsilon.$$

Theorem 1.1. Suppose $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{R}^n$. Then (1) holds if and only if

$$\lim_{t \to a} r_i(t) = l_i \text{ for } i \in \{1, \ldots, n\}.$$

Definition 1.2. We say $\mathbf{r}$ is differentiable at $a$ if $a$ is an interior point of $I$ and there is $\mathbf{r}'(a) \in \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{\mathbf{r}(a + h) - \mathbf{r}(a)}{h} = \mathbf{r}'(a).$$

Theorem 1.2. $\mathbf{r}$ is differentiable at $a$ if and only if $r_i$ is differentiable at $a$ for each $i \in \{1, \ldots, n\}$ in which case we have

$$\mathbf{r}'(a) = (r'_1(a), \ldots, r'_n(a)).$$

2. Limit and Differentiation Rules.

In 12.5 there ought to be limit rules following the pattern of Theorem 2. Let me illustrate by example. Throughout this section $I$ is an open interval, $a \in I$ and

$$\mathbf{u}, \mathbf{v} : I \to \mathbb{R}^3.$$

Theorem 2.1. Suppose $\lim_{t \to a} \mathbf{u}(t) = \mathbf{b}$ and $\lim_{t \to a} \mathbf{v}(t) = \mathbf{c}$. Then

$$\lim_{t \to a} \mathbf{u} \times \mathbf{v}(t) = \mathbf{b} \times \mathbf{c}.$$

Proof. Suppose $t \in I$. Then

$$|\mathbf{u} \times \mathbf{v}(t) - \mathbf{u} \times \mathbf{v}(a)| = |(\mathbf{u}(t) - \mathbf{u}(a)) \times \mathbf{v}(a) + \mathbf{u}(t) \times (\mathbf{v}(t) - \mathbf{v}(a))| \leq |(\mathbf{u}(t) - \mathbf{u}(a)) \times \mathbf{v}(a)|| + |\mathbf{u}(t) \times (\mathbf{v}(t) - \mathbf{v}(a))| \leq |\mathbf{u}(t) - \mathbf{u}(a)||\mathbf{v}(a)|| + |\mathbf{u}(t)|, |\mathbf{v}(t) - \mathbf{v}(a)||;$$

this last quantity approaches 0 as $t$ approaches $a$ by limit rules from one variable calculus; here we have used the triangle inequality and the fact that the length of a cross product does not exceed the product of the length of the factors. $\square$
Theorem 2.2. Suppose \( u \) and \( v \) are differentiable at \( a \). Then \( u \times v \) is differentiable at \( a \) and
\[
(u \times v)'(a) = u'(a) \times v(a) + u(a) \times v'(a).
\]

Proof. Suppose \( h \in \mathbb{R} \) and \( a + h \in I \). Then
\[
\frac{1}{h} ((u \times v)(a + h) - (u \times v)(a)) = \left( \frac{u(a + h) - u(a)}{h} \right) \times v(a) + u(a + h) \times \left( \frac{v(a + h) - v(a)}{h} \right);
\]
now apply limit rules to obtain the desired result. \( \square \)

3. Velocity, acceleration and speed.

Definition 3.1. Suppose \( r \) is a curve. Then
\[
r' \text{ is its velocity,}
\]
\[
r'' \text{ is its acceleration}
\]
and
\[
|r'| \text{ is its speed.}
\]
(Many velocity and acceleration are vectors and speed is a scalar. Forgetting this leads to all sorts of confusion.)

4. Integration.

Suppose \( r \) is a curve. Then its integral
\[
\int_a^b r(t) \, dt
\]
can be defined using Riemann sums in the same way one defines the integral of a scalar. Note the stuff on pages 809-811.

5. Projectile motion.

Suppose \( r \) is the path of a projectile with mass \( m \) which is subject to the force \(-gk\) where \( g \) is the gravitational constant for the Earth’s surface in units consistent with those of \( m \). Then Newton’s Second Law of Motion says
\[
(mr')' = -gk.
\]
If the mass is constant this becomes
\[
mr'' = -gk.
\]
Let \( t_0 \in I \), let
\[
r_0 = r(t_0) \quad \text{and let} \quad v_0 = r'(t_0).
\]
(That is, \( r_0 \) and \( v_0 \) are the initial position and velocity, respectively. Integrating from \( t_0 \) to \( t \) we obtain
\[
m(r'(t) - v_0) = -g(t - t_0)k
\]
so
\[
r'(t) = v_0 - \frac{g}{m}(t - t_0)k.
\]
Integrating one more time from \( t_0 \) to \( t \) we obtain
\[
r(t) - r_0 = (t - t_0)v_0 - \frac{g}{2m}(t - t_0)^2k
\]
\[ r(t) = r_0 + (t - t_0)v_0 - \frac{g}{2m}(t - t_0)^2 k. \]

In particular, the range of \( r \) lies in any plane containing the initial position, the initial velocity and \( k \).

6. **Problem 62 on page 815.**

As written it makes no sense. What they probably mean is that if in Newton’s Second Law

\[ F = ma \]

(constant mass) where we are moving in \( \mathbb{R}^3 \) we have

\[ F \parallel r \]

then the range (or track in the book) of \( r \) lies in a plane. This is very interesting and useful. It’s why, for example, the planetary motion, in the two body version, lies in a plane containing the Sun.