1. Arclength reparameterization.

Suppose I is an interval and

$$\mathbf{r} : I \to \mathbb{R}^n$$

is a curve in $\mathbb{R}^n$ whose speed is never zero. Suppose $t_0 \in I$ and let

$$\sigma(t) = \int_{t_0}^{t} |\mathbf{v}|(\tau)\, d\tau \quad \text{for } \tau \in I.$$

Then σ is strictly increasing with range some interval H and

$$\sigma'(t) = |\mathbf{v}|(t) \quad \text{for } t \in I.$$

Let

$$\phi : H \to I$$

be the function which is inverse to σ. Then

$$\phi(\sigma(t)) = t \quad \text{for } t \in I \text{ and } \quad \sigma(\phi(s)) = s \quad \text{for } s \in H.$$

From the chain rule we obtain

$$\sigma'(t) = \frac{1}{\phi'(\sigma(t))} \quad \text{for } t \in I \quad \text{and} \quad \phi'(s) = \frac{1}{\sigma'(\phi(s))} \quad \text{for } s \in H.$$

Let

$$\mathbf{q}(s) = \mathbf{r}(\phi(s)) \quad \text{for } s \in H.$$

I claim that $\mathbf{q}$ has unit speed; it is called an **arclength reparameterization of r.** Indeed, by the chain rule,

$$|\mathbf{q}'(s)| = |\phi'(s)\mathbf{r}'(\phi(s))| = \left| \frac{1}{\sigma'(\phi(s))} \mathbf{r}'(\phi(s)) \right| = 1,$$

as desired.

2. Curvature and other neat stuff.

Suppose I is an interval in $\mathbb{R}$ and

$$\mathbf{r} : I \to \mathbb{R}^n$$

is a (parametric) curve in $\mathbb{R}^n$. We have already defined speed, velocity and acceleration. Suppose the speed $|\mathbf{v}|$ never vanishes. Let

$$\mathbf{T} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

and let

$$\mathbf{K} = \frac{1}{|\mathbf{v}|}\mathbf{T}'.$$

These vector functions are called the **unit tangent** and **curvature vector** of $\mathbf{r}$, respectively. Let

$$\kappa = |\mathbf{K}|;$$

this nonnegative scalar function is called the **curvature** of $\mathbf{r}$.

Now suppose $\kappa > 0$. Let

$$\mathbf{N} = \frac{1}{\kappa}\mathbf{K}$$

which is obviously equivalent to

$$\frac{1}{|\mathbf{v}|}\mathbf{T}' = \kappa\mathbf{N}.$$

Note that

$$|\mathbf{v}|' = |\mathbf{r}'|' = \frac{\mathbf{r}'' \bullet \mathbf{r}'}{|\mathbf{r}'|} = \frac{\mathbf{a} \bullet \mathbf{v}}{|\mathbf{v}|}.$$

Differentiation $\mathbf{r}' = |\mathbf{v}|\mathbf{T}$ we find that

$$\begin{aligned} \mathbf{a} = \mathbf{r}'' &= |\mathbf{v}|'\mathbf{T} + |\mathbf{v}|\mathbf{T}' \\ &= \frac{\mathbf{a} \bullet \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} + |\mathbf{v}|^2\mathbf{K} \\ &= \mathbf{comp}_{\mathbf{v}}\mathbf{a} + |\mathbf{v}|^2\mathbf{K} \end{aligned}$$

so

$$\mathbf{K} = \frac{1}{|\mathbf{v}|^2}\left(\mathbf{a} - \mathbf{comp}_{\mathbf{v}}\mathbf{a}\right)$$

and

$$\mathbf{a} = \mathbf{comp}_{\mathbf{v}}\mathbf{a} + \kappa|\mathbf{v}|^2\mathbf{N}.$$

A simple computation gives

$$\kappa = |\mathbf{K}| = \frac{\sqrt{|\mathbf{a}|^2|\mathbf{v}|^2 - (\mathbf{a} \bullet \mathbf{v})^2}}{|\mathbf{v}|^3}.$$

Since

$$\mathbf{a} \bullet \mathbf{v} = \frac{1}{2}|\left(\mathbf{v}|^2\right)' = |\mathbf{v}|\,|\mathbf{v}|'$$

we find that

$$\mathbf{a} = |\mathbf{v}|'\mathbf{T} + \kappa|\mathbf{v}|^2\mathbf{N}.$$

The interesting thing about $\mathbf{K}$, κ and $\mathbf{N}$ is that they depend only on the range of $\mathbf{r}$; in other words, they are *independent of parameterization*. This means, by definition, that if

$$\phi : H \to I$$

is twice continuously differentiable and strictly increasing or decreasing with range equal I, if

$$\mathbf{q}(s) = \mathbf{r}(\phi(s)) \quad \text{for } s \in H$$

and if $\mathbf{J}$ is the curvature vector of $\mathbf{q}$ then

$$\mathbf{J}(s) = \mathbf{K}(\phi(s)) \quad \text{for } s \in H. \tag{1}$$

This immediately implies that the normal vector at s of $\mathbf{q}$ equals the normal vector at $\phi(s)$ of $\mathbf{r}$. Indeed, by the chain rule we find that

$$\mathbf{q}'(s) = \phi'(s)\mathbf{r}'(\phi(s));$$

in particular,

$$\mathbf{comp}_{\mathbf{q}'(s)}\mathbf{x} = \mathbf{comp}_{\mathbf{v}(\phi(s))}\mathbf{x} \quad \text{for } \mathbf{x} \in \mathbb{R}^3. \tag{2}$$

By Leibniz' rule and the chain rule, we have

$$\mathbf{q}''(s) = \phi''(s)\mathbf{r}'(\phi(s)) + (\phi'(s))^2\mathbf{r}''(\phi(s)) = \phi''(s)\mathbf{v}(\phi(s)) + (\phi'(s))^2\mathbf{a}(\phi(s));$$

keeping in mind (2) we find that

$$\mathbf{comp}_{\mathbf{q}'(s)}\mathbf{q}''s) = \phi''(s)\mathbf{v}(\phi(s)) + (\phi'(s))^2\mathbf{comp}_{\mathbf{v}(\phi(s))}\mathbf{a}(\phi(s)),$$

thereby establishing (1).

3. The binormal and torsion.

Let $\mathbf{r}$ be a curve in $\mathbb{R}^3$ parameterized by arclength. Let $\mathbf{T}$ be its velocity and let $\mathbf{N}$ be its normal. Let

$$\mathbf{B} = \mathbf{T} \times \mathbf{N};$$

this vector (function) is call the **binormal**. Note that $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are mutually perpendicular unit vectors such that

$$[\mathbf{T}, \mathbf{N}, \mathbf{B}] = 1.$$

Let τ be the scalar function determined the requirement that

$$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B};$$

τ is called the **torsion**. (Question: Why does this work? Answer: Because the matrix

$$\begin{bmatrix} \mathbf{N}' \bullet \mathbf{T} & \mathbf{N}' \bullet \mathbf{N} & \mathbf{N}' \bullet \mathbf{B} \\ \mathbf{T}' \bullet \mathbf{T} & \mathbf{T}' \bullet \mathbf{N} & \mathbf{T}' \bullet \mathbf{B} \\ \mathbf{B}' \bullet \mathbf{T} & \mathbf{B}' \bullet \mathbf{N} & \mathbf{B}' \bullet \mathbf{B} \end{bmatrix}$$

is skewsymmetric.

It follows that

$$\mathbf{B}' = -\tau\mathbf{N}.$$

In matrices we have

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

This leads to the following.

Theorem 3.1. $\mathbf{r}$ lies in a plane if and only $\tau = 0$.

Proof. $\tau = 0$ if and only if $\mathbf{B}$ is constant, say $\mathbf{b}$ in which case $\mathbf{T}$ lies in the plane $P = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \bullet \mathbf{b} = 0\}$. Now for any t in the domain of $\mathbf{r}$ we have

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{T}(\tau)\, d\tau$$

which lies in the plane $\mathbf{r}(t_0) + P$. □

Theorem 3.2. $\mathbf{r}$ lies in a circle if and only if $\tau = 0$ and κ is constant.

Proof. Suppose $\tau = 0$ and κ is constant. From the preceding Theorem we know that the range of $\mathbf{r}$ lies in a plane P. Moreover,

$$\left(\mathbf{r} + \frac{1}{\kappa}\mathbf{N}\right)' = \kappa\mathbf{T} + \frac{1}{\kappa}(-\kappa\mathbf{T}) = \mathbf{0}$$

so there is a constant vector $\mathbf{c}$ such that

$$\mathbf{r} + \frac{1}{\kappa}\mathbf{T} = \mathbf{c}.$$

That is, $|\mathbf{r} - \mathbf{c}| = 1/\kappa$ so $\mathbf{r}$ lies in the circle in P with center $\mathbf{c}$ and radius $1/\kappa$. □

Remark 3.1. It's not too hard to show that given an interval I, a positive function $\kappa : I \to \mathbb{R}$ and a function $\tau : I \to \mathbb{R}$ there is a curve in space with curvature κ and torsion τ; moreover, if two curves have the same curvature and torsion one is a rigid motion applied to the other.

Now fix a point s_0 in the domain of $\mathbf{r}$. From Taylor's Theorem we have

(3)
$$\mathbf{r}(s) = \mathbf{r}(s_0) + (s - s_0)\mathbf{r}'(s_0) + \frac{(s-s_0)^2}{2}\mathbf{r}''(s_0) + \frac{(s-s_0)^3}{6}\mathbf{r}'''(s_0) + O(|s-s_0|^4).$$

Now

(4)
$$\begin{aligned}
\mathbf{r}' &= \mathbf{T}; \\
\mathbf{r}'' = \mathbf{T}' &= \kappa\mathbf{N}; \\
\mathbf{r}''' &= (\kappa\mathbf{N})' \\
&= \kappa'\mathbf{N} + \kappa\mathbf{N}' \\
&= -\kappa^2\mathbf{T} + \kappa'\mathbf{N} + \kappa\tau\mathbf{B};
\end{aligned}$$

evaluating at s_0 and substituting back in (3) we obtain

$$\begin{aligned}
\mathbf{r}(s) = \mathbf{r}(s_0) \\
&+ (s-s_0)\mathbf{T}(s_0) \\
&+ \frac{(s-s_0)^2}{2}\kappa(s_0)\mathbf{N}(s_0) \\
&+ \frac{(s-s_0)^3}{6}(-\kappa(s_0)^2\mathbf{T}(s_0) + \kappa'(s_0)\mathbf{N}(s_0) \\
&+ \kappa(s_0)\tau(s_0)\mathbf{B}(s_0)) \\
&+ O(|s-s_0|^4) \\
&= \mathbf{r}(s_0) \\
&+ \left((s-s_0) - \kappa(s_0)^2\frac{(s-s_0)^3}{6}\right)\mathbf{T}(s_0) \\
&+ \left(\kappa(s_0)\frac{(s-s_0)^2}{2} + \kappa'(s_0)\frac{(s-s_0)^3}{6}\right)\mathbf{N}(s_0) \\
&+ \left(\kappa(s_0)\tau(s_0)\frac{(s-s_0)^3}{6}\right)\mathbf{B}(s_0)) \\
&+ O(|s-s_0|^4).
\end{aligned}$$