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A BOUNDARY APPROXIMATION ALGORITHM FOR PLANAR DOMAINS

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Contents

1. Caveat.

In what follows no hypotheses has been made on the triangulation $\mathcal{T}$. As relatively simple counterexamples show, the results in the paper do not hold as stated. I have fixed all this and a newer, correct version will appear shortly.

2. Introduction.

Suppose $\mathcal{T}$ is a triangulation $\mathcal{T}$ of $\mathbb{R}^2$ with edges $\mathcal{E}$ and vertices $\mathcal{V}$, and suppose Ω is a bounded open subset of $\mathbb{R}^2$. Let

$$\mathcal{V}_{\text{in}} = \{v \in \mathcal{V} : v \in \Omega\}.$$

In this paper we will give an algorithm which uses $\mathcal{V}_{\text{in}}$ and nothing else to construct a finite disjointed family $\mathcal{P}$ of simple closed polygons whose union approximates, in a sense we shall make precise, the boundary $\partial\Omega$ of Ω. Of course we will need to assume something about Ω; under our assumptions $\partial\Omega$ will have a finitely many connected components equal in number to the number of members of $\mathcal{P}$. Specifically, we assume that $\partial\Omega$ is continuously differentiable and *and* that, for some positive real number R, if $a \in \mathbb{R}^2$ and $\mathbf{dist}\,(a, \partial\Omega) < R$ there is a *unique* point of $\partial\Omega$ closest to a. That is, $\partial\Omega$ has **reach** R in the sense of [**?**]. If $\partial\Omega$ is *twice* continuously differentiable it will have positive reach R and if K is the maximum length of the curvature vector of $\partial\Omega$ we have $K \leq 1/R$; however, KR can be arbitrarily small. In particular, R is a constraint on how much $\partial\Omega$ can come back on itself.

We will show that if

$$h = \sup\{\mathbf{diam}\, T : T \in \mathcal{T}\} < R$$

then

$$\mathbf{length}\,(P) \leq \mathbf{length}\,(\partial\Omega) \leq \frac{R}{R-h}\mathbf{length}\,(P)$$

where $P = \cup\mathcal{P}$.

Let $\mathcal{E}_{\text{bdry}}$ be the set of $E \in \mathcal{E}$ such that one vertex of E lies in $\mathcal{V}_{\text{in}}$ and the other does not. Let α, the basic **adjacency relation**, be the set of $(E, F) \in \mathcal{E}_{\text{bdry}} \times \mathcal{E}_{\text{bdry}}$ such that $E \neq F$ and E and F have a common vertex and let N be its cardinality. Our algorithm runs in time $O(N^2)$ given α. *However, if* $\mathcal{V} = A[\mathbb{Z}^2]$ *for some affine*

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isomorphism of $\mathbb{R}^2$ *then our algorithm runs in time* $O(N)$ *with a small constant and uses only integer arithmetic.*

This algorithm has been implemented on the computer and has been run in situations where $\partial\Omega$ is quite irregular. It turns out that P, even in these cases, appears to be a sparse approximation to $\partial\Omega$, and so may useful even when $\partial\Omega$ does not have positive reach.

As we shall show, $\mathcal{P}$ is an *affine* invariant of $\mathcal{V}_{\text{in}}$ even though the length of P clearly is not. It also turns out $\mathcal{P}$ is uniquely determined by $\mathcal{V}_{\text{in}}$.

3. Preliminaries.

We let

$$\mathbb{N} \quad \text{and} \quad \mathbb{N}^+$$

be the set of nonnegative integers and the set of positive integers, respectively. Whenever $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ we let

$$\mathbb{I}(m,n) = \{i \in \mathbb{Z} : m \leq i < m+n\}.$$

Whenever f is a function whose domain is a subset of $\mathbb{Z}$ and z is in its domain we will write f_z instead of $f(z)$.

Whenever r is a relation and A is a set we let

$$r[A] = \{y : \text{for some } x,\ x \in A \text{ and } (x,y) \in r\}.$$

In particular, if X is a set and f is a function with domain X then

$$f[A] = \{f(x) : x \in X \cap A\}.$$

We let

$$\mathbb{R}_2$$

be the dual of $\mathbb{R}^2$.

Let

$$\mathbf{e}_1 = (1,0) \quad \text{and let} \quad \mathbf{e}_2 = (0,1);$$

thus $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard basis vectors for $\mathbb{R}^2$.

We let

$$a^\perp = (a_1, -a_2) \quad \text{whenever } a = (a_1, a_2) \in \mathbb{R}^2.$$

We let

$$a \times b = a^\perp \bullet b = -a \bullet b^\perp \quad \text{whenever } a, b \in \mathbb{R}^2.$$

Alternatively, if $a = (a_1, a_2) \in \mathbb{R}^2$ and $b = (b_1, b_2) \in \mathbb{R}^2$ then

$$a \times b = \mathbf{det} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

Whenever $a, b, c \in \mathbb{R}^2$ we let

$$[a,b,c] = (b-a) \times (c-a) = a \times b + b \times c + c \times a;$$

note that $[a,b,c] \neq 0$ if and only if the points a, b, c are noncollinear as well as that $[\pi(a), \pi(b), \pi(c)] = \sigma[a,b,c]$ whenever π is a permutation of $\{a,b,c\}$ and σ is the index of π.

We let

$$\mathbf{H}$$

be the family of closed halfspaces in $\mathbb{R}^2$; thus $H \in \mathbf{H}$ if and only if for some $\omega \in \mathbb{R}_2$ and $z \in \mathbb{R}$ we have $H = \{x \in \mathbb{R}^2 : \omega(x) \le z\}$. For $a, b \in \mathbb{R}^2$ with $a \neq b$ we let

$$\mathbf{h}_+(a,b) = \{x \in \mathbb{R}^2 : [a,b,x] \ge 0\} \in \mathbf{H}$$

and we let

$$\mathbf{h}_-(a,b) = \{x \in \mathbb{R}^2 : [a,b,x] \le 0\} \in \mathbf{H}.$$

Given $N \in \mathbb{N}^+$ and $a_1, \ldots, a_N \in \mathbb{R}^2$ we let

$$\mathbf{c}(a_1, \ldots, a_N) \quad \text{be the convex hull of } \{a_1, \ldots, a_N\}.$$

We let

$$\begin{aligned} \mathbf{V} &= \{a : a \in \mathbb{R}^2\}; \\ \mathbf{E} &= \{\mathbf{c}(a,b) : a, b \in \mathbf{V} \text{ and } a \neq b\}; \\ \mathbf{T} &= \{\mathbf{c}(a,b,c) : a,b,c \in \mathbf{V} \text{ and } [a,b,c] \neq 0\}. \end{aligned}$$

For $E \in \mathbf{E}$ we let $\mathbf{v}(E) = \{a, b\}$ where a, b are such that $E = \mathbf{c}(a,b)$; the members of $\mathbf{v}(E)$ are called **vertices of** E.

For $T \in \mathbf{T}$ we let $\mathbf{e}(T) = \{\mathbf{c}(\{a,b\}), \mathbf{c}(\{b,c\}), \mathbf{c}(\{c,a\})\}$ and we let $\mathbf{v}(T) = \{a,b,c\}$ where a,b,c are such that $T = \mathbf{c}(\{a,b,c\})$; the members of $\mathbf{e}(T)$ are called **edges of** T and the members of $\mathbf{v}(T)$ are called **vertices of** T.

4. The triangulation and the set of vertices $\mathcal{V}_{\text{in}}$.

For the remainder of this paper we fix a triangulation

$$\mathcal{T}$$

of $\mathbb{R}^2$; this means, by definition, that

(i) $\mathcal{T} \subset \mathbf{T}$;
(ii) $\mathbb{R}^2 = \cup \mathcal{T}$;
(iii) if $T, U \in \mathcal{T}$, $T \neq U$ and $T \cap U \neq \emptyset$ then either there is $E \in \mathbf{e}(T) \cap \mathbf{e}(U)$ such that $T \cap U = E$ or there is $v \in \mathbf{v}(T) \cap \mathbf{v}(U)$ such that $T \cap U = \{v\}$;
(iv) $\{T \in \mathcal{T} : T \cap K \neq \emptyset\}$ if finite whenever K is a compact subset of $\mathbb{R}^2$.

We let

$$\mathcal{E} = \{E : E \in \mathbf{e}(T) \text{ for some } T \in \mathcal{T}\}$$

and we let

$$\mathcal{V} = \cup\{v : v \in \mathbf{v}(T) \text{ for some } T \in \mathcal{T}\}.$$

For the remainder of this paper we fix nonempty subsets

$$\mathcal{V}_{\text{in}} \quad \text{and} \quad \mathcal{V}_{\text{out}}$$

such that

$$\mathcal{V} = \mathcal{V}_{\text{in}} \cup \mathcal{V}_{\text{out}}, \quad \mathcal{V}_{\text{in}} \cap \mathcal{V}_{\text{out}} = \emptyset \quad \textit{and } \mathcal{V}_{\text{in}} \textit{ is finite.}$$

We let

$$\begin{aligned} \mathcal{E}_{\text{in}} &= \{E \in \mathcal{E} : \mathbf{v}(E) \subset \mathcal{V}_{\text{in}}\}; \\ \mathcal{E}_{\text{out}} &= \{E \in \mathcal{E} : \mathbf{v}(E) \subset \mathcal{V}_{\text{out}}\}; \\ \mathcal{E}_{\text{bdry}} &= \mathcal{E} \sim (\mathcal{E}_{\text{in}} \cup \mathcal{E}_{\text{out}}); \\ \mathcal{T}_{\text{out}} &= \{T \in \mathcal{T} : \mathbf{v}(T) \subset \mathcal{V}_{\text{out}}\}; \\ \mathcal{T}_{\text{in}} &= \{T \in \mathcal{T} : \mathbf{v}(T) \subset \mathcal{V}_{\text{in}}\}; \\ \mathcal{T}_{\text{bdry}} &= \mathcal{T} \sim (\mathcal{T}_{\text{in}} \cup \mathcal{T}_{\text{out}}) \end{aligned}$$

and we note that all of these sets are finite. We let

$$\mathcal{V}_{\mathrm{bdry}} = \{v \in \mathcal{V} : \text{for some } E \in \mathcal{E}_{\mathrm{bdry}},\, v \in \mathbf{v}(E)\}.$$

We define

$$\mathbf{v}_{\mathrm{in}} : \mathcal{E}_{\mathrm{bdry}} \to \mathcal{V}_{\mathrm{in}} \quad \text{and} \quad \mathbf{v}_{\mathrm{out}} : \mathcal{E}_{\mathrm{bdry}} \to \mathcal{V}_{\mathrm{out}}$$

by requiring that $\mathbf{v}_{\mathrm{in}}(E) \in \mathcal{V}_{\mathrm{in}}$, $\mathbf{v}_{\mathrm{out}}(E) \in \mathcal{V}_{\mathrm{out}}$ and $\mathbf{v}(E) = \{\mathbf{v}_{\mathrm{out}}(E), \mathbf{v}_{\mathrm{in}}(E)\}$ for each $E \in \mathcal{E}_{\mathrm{bdry}}$.

4.1. **The adjacency relation α and the permutation σ.** We have the basic adjacency relation α is defined as follows.

Definition 4.1. We let

$$\alpha = \{(E,F) \in \mathcal{E}_{\mathrm{bdry}} \times \mathcal{E}_{\mathrm{bdry}} : \{E,F\} \subset \mathbf{e}(T) \text{ for some } T \in \mathcal{T}_{\mathrm{bdry}}\}.$$

The following Proposition is a direct consequence of the definitions.

Proposition 4.1. If $T \in \mathcal{T}_{\mathrm{bdry}}$ then exactly two edges of T belong to $\mathcal{E}_{\mathrm{bdry}}$.

Definition 4.2. Whenever $I, J \in \mathbb{Z}$ we let

$$\mathcal{C}(I,J)$$

be the set of maps $E : \mathbb{I}(I,J) \to \mathcal{E}_{\mathrm{bdry}}$ such that

(i) $(E_i, E_{i+1}) \in \alpha$ whenever $\{i, i+1\} \subset \mathbb{I}(I,J)$;
(ii) $E_i \neq E_{i+2}$ whenever $\{i, i+1, i+2\} \subset \mathbb{I}(I,J)$.

We say a subset $\mathcal{F}$ of $\mathcal{E}_{\mathrm{bdry}}$ is **connected** if it equals the range of a chain.

Definition 4.3. Suppose $v \in \mathcal{V}_{\mathrm{bdry}}$. We let

$$\mathcal{S}(v) = \{E \in \mathcal{E}_{\mathrm{bdry}} : v \in \mathbf{v}(E)\}$$

and we let

$$\mathbf{S}(v)$$

be the collection of maximal connected subsets of $\mathcal{S}(v)$. For each $\mathcal{F} \in \mathbf{S}(v)$ we let

$$\mathbf{B}(v, \mathcal{F})$$

be the set of $E \in \mathcal{E}_{\mathrm{bdry}}$ such that $v \notin \mathbf{v}(E)$ and $\{E\} \cup \mathcal{F}$ is connected.

Definition 4.4. We say $\gamma \in \Gamma$ is **special** if the following conditions hold:

(I) if $(D,E,F) \in \tau$ and $\gamma(E) \in E \sim \mathbf{v}(E)$ then $\{\gamma(D), \gamma(E), \gamma(F)\}$ is linear;
(II) if $v \in \mathcal{V}_{\mathrm{bdry}}$, $\mathcal{F} \in \mathbf{S}(v)$ and $v \in \{\gamma(F) : F \in \mathcal{F}\}$ then
  (a) $\gamma(F) = v$ for all $F \in \mathcal{F}$;
  (b) if $\{D, E\} = \mathbf{B}(v, \mathcal{F})$ then

$$F \cap \mathbf{w}_v(\mathbf{c}(\{\gamma(E), \gamma(F)\}) = \emptyset \quad \text{whenver } F \in \mathcal{F}.$$

Proposition 4.2. Suppose $v \in \mathcal{V}_{\mathrm{bdry}}$ and $\mathcal{F} \in \mathbf{S}(v)$. Then exactly one of the following statements holds:

(i) $\mathbf{card}\, \mathbf{B}(v, \mathcal{F}) = 2$;
(ii) $\mathbf{card}\, \mathbf{B}(v, \mathcal{F}) = 0$ and $\mathcal{S}(v) = \mathcal{F}$.

Proposition 4.3. Suppose $v, \mathcal{F}$ and v are as in 4.13 (II), E, F are such that $\mathbf{B}(v, \mathcal{F}) = \{E, F\}$, and $I \in \mathbb{N}^+$ is such that $F = \sigma^I[E]$. Then

$$(\gamma(F) - v) \times (v - \gamma(E)) \begin{cases} \leq 0 & \text{if } v \in \mathcal{V}_{\mathrm{out}}, \\ \geq 0 & \text{if } v \in \mathcal{V}_{\mathrm{in}}. \end{cases}$$

The permutation σ which we now define will be useful in what follows.

Definition 4.5. For each $E \in \mathcal{E}_{\rm bdry}$ we let

$$\mathbf{j}_+(E) = \mathbf{h}_+(\mathbf{v}_{\rm in}(E), \mathbf{v}_{\rm out}(E)) \quad \text{and we let} \quad \mathbf{j}_-(E) = \mathbf{h}_-(\mathbf{v}_{\rm in}(E), \mathbf{v}_{\rm out}(E)).$$

We let

$$\sigma = \{(E, F) \in \alpha : F \subset \mathbf{j}_+(E)\}.$$

Proposition 4.4. σ is a permutation of $\mathcal{E}_{\rm bdry}$ without fixed points and

$$\alpha = \sigma \cup \sigma^{-1}.$$

Proof. It follows directly from Proposition 4.1 that σ and σ^{-1} are functions which are inverse to each other. It is obvious that $\alpha = \sigma \cup \sigma^{-1}$. □

Definition 4.6. We let

$$\mathbf{O}$$

be the set of orbits of the action

$$\mathbb{Z} \times \mathcal{E}_{\rm bdry} \ni (i, E) \mapsto \sigma^i[E] \in \mathcal{E}_{\rm bdry}$$

of $\mathbb{Z}$ on $\mathcal{E}_{\rm bdry}$. For each $E \in \mathcal{E}_{\rm bdry}$ we let

$$\mathbf{o}(E) = \{\sigma^i[E] : n \in \mathbb{Z}\};$$

thus $\mathbf{o}(E)$ is the orbit of E under the aforementioned action.

The following two Proposition should be evident.

Proposition 4.5. For any $E \in \mathcal{E}_{\rm bdry}$ we have

$$\mathbf{o}(E) = \{\sigma^n[E] : n \in \mathbb{Z}\}.$$

Moreover, if $E, F \in \mathcal{O} \in \mathbf{O}$ there is one and only one $i \in \mathbb{Z}$ such that $0 \le i < \mathbf{card}\,\mathcal{O}$ and $F = \sigma^i[F]$.

Proposition 4.6. Z has $\mathbf{card}\,\mathbf{O}$ connected components each of which is homeomorphic to a circle.

Definition 4.7. Suppose $E \in \mathcal{E}_{\rm bdry}$. We let the **order of** E equal $\min\{n \in \mathbb{N}^+ : E = \sigma^n[E]\}$. We say E is **degenerate** if $\mathbf{v}(E) \cap \mathbf{v}(F) \neq \emptyset$ whenever $F \in \mathbf{o}(E)$.

We leave the straightforward proofs of the following two Propositions to the reader.

Proposition 4.7. Suppose $E \in \mathcal{E}_{\rm bdry}$. Then $\mathbf{o}(E)$ has at least three members.

Proposition 4.8. Suppose $E \in \mathcal{E}_{\rm bdry}$ and E is degenerate. Then there is $v \in \mathcal{V}$ such that $\cap\{\mathbf{v}(F) : F \in \mathbf{o}(E)\} = \{v\}$.

4.2. **The family Γ.**

Definition 4.8. We let

$$\Gamma$$

be the set of functions $\gamma : \mathcal{E}_{\rm bdry} \to \mathbf{V}$ such that $\gamma(E) \in E$ whenever $E \in \mathcal{E}_{\rm bdry}$; in other words, Γ is the set of choice functions for $\mathcal{E}_{\rm bdry}$.

Keeping in mind Proposition 4.1, for each $(\gamma, T) \in \Gamma \times \mathcal{T}_{\rm bdry}$ we let

$$\mathbf{p}(\gamma, T) = \mathbf{c}(\gamma(E), \gamma(F)) \quad \text{and} \quad \mathbf{l}(\gamma, T) = |\gamma(E) - \gamma(F)| = \mathbf{diam}\,\mathbf{p}(\gamma, T)$$

where $\{E, F\} = \mathcal{E}_{\rm bdry} \cap \mathbf{e}(T)$.

For each $\gamma \in \Gamma$ we let

$$\mathbf{p}(\gamma) = \cup_{T \in \mathcal{T}_{\rm bdry}} \mathbf{p}(\gamma, T) \quad \text{and we let} \quad \mathbf{l}(\gamma) = \sum_{T \in \mathcal{T}_{\rm bdry}} \mathbf{l}(\gamma, T).$$

Definition 4.9. For each $t \in [0,1]$ we define

$$\mu_t \in \Gamma$$

by letting $\mu_t(E) = (1-t)\mathbf{v}_{\rm in}(E) + t\mathbf{v}_{\rm out}(E)$ whenever $E \in \mathcal{E}_{\rm bdry}$, we let

$$Z_t = \mathbf{p}(\mu_t).$$

The following Proposition should be clear.

Proposition 4.9. Suppose $t \in (0,1)$. Then Z_t has finitely many connected components each of which is a simple closed polygon and the number of which equals $\mathbf{card}\,\mathbf{O}$.

Definition 4.10. Suppose $\gamma_i \in \Gamma$, $i = 1, 2$. We say γ_1 **is equivalent to** γ_2 and write $\gamma_1 \approx \gamma_2$ if

$$E \in \mathcal{E}_{\rm bdry} \text{ and } \gamma_1(E) \neq \gamma_2(E) \;\Rightarrow\; \gamma_1(D) = \gamma_2(D) \quad \text{whenever } (D, E) \in \alpha.$$

The following Proposition should be clear.

Proposition 4.10. Suppose $\gamma_i \in \Gamma$, $i = 1, 2$, and $\gamma_1 \approx \gamma_2$. Then $\mathbf{p}(\gamma_1) = \mathbf{p}(\gamma_2)$ and $\mathbf{l}(\gamma_1) = \mathbf{l}(\gamma_2)$.

4.3. **If $\mathbf{p}(\gamma_1) = \mathbf{p}(\gamma_2)$.** We show in Theorem 4.1 below that if $\gamma_i \in \Gamma$, $i \in \{1, 2\}$ and $\mathbf{p}(\gamma_1) = \mathbf{p}(\gamma_2)$ then γ_1 and γ_2 are essentially the same.

Lemma 4.1. Suppose $\mathcal{U} \subset \mathcal{T}_{\rm bdry}$, $U = \cup\mathcal{U}$ and $\gamma \in \Gamma$. Then

$$\mathbf{p}(\gamma) \cap U = \cup\{\mathbf{p}(\gamma, T) \cap U : U \in \mathcal{U}\}.$$

Proof. This follows directly from the fact that if $V \in \mathcal{T}$ then $V \cap U \neq \emptyset$ if and only if $V \in \mathcal{U}$. □

The next three Lemmas are geometrically obvious; we leave their proofs to the reader.

Lemma 4.2. Suppose $\gamma \in \Gamma$, $T \in \mathcal{T}_{\rm bdry}$ and $\{E, F\} = \mathbf{e}(T) \cap \mathcal{E}_{\rm bdry}$. Then $\mathbf{p}(\gamma) \cap \mathbf{int}\, T$ is nonempty if and only if *either* $\gamma(E) \notin \mathbf{v}(E)$ and $\gamma(F) \notin E$ *or* $\gamma(F) \notin \mathbf{v}(E)$ and $\gamma(E) \notin F$.

Lemma 4.3. Suppose $\gamma_i \in \Gamma$, $i = 1, 2$; $T \in \mathcal{T}_{\rm bdry}$; and $\mathbf{p}(\gamma_1) \cap \mathbf{int}\, T$ and $\mathbf{p}(\gamma_2) \cap \mathbf{int}\, T$ are equal and nonempty. Then $\gamma_1(E) = \gamma_2(E)$ whenever $E \in \mathbf{e}(T) \cap \mathcal{E}_{\rm bdry}$.

Lemma 4.4. Suppose $\gamma \in \Gamma$; $E \in \mathcal{E} \sim \mathcal{E}_{\rm bdry}$ and

$$\mathbf{p}(\gamma) \cap (E \sim \mathbf{v}(E)) \neq \emptyset.$$

Then there is one and only $T \in \mathcal{T}_{\rm bdry}$ such that $E \in \mathbf{e}(T)$ and if D, F are such that $\mathbf{e}(T) = \{D, E, F\}$ then

$$\mathbf{v}(E) = \{\gamma(D), \gamma(F)\}.$$

Theorem 4.1. Suppose $\gamma_i \in \Gamma$ for $i \in \{1, 2\}$ and $\mathbf{p}(\gamma_1) = \mathbf{p}(\gamma_2)$. Then $\gamma_1 \approx \gamma_2$. Moreover, if $E \in \mathcal{E}_{\rm bdry}$ and $\gamma_1(E) \neq \gamma_2(E)$ then

$$\mathbf{v}(D) \cap \mathbf{v}(E) \cap \mathbf{v}(F) = \emptyset$$

where $\{D, F\} = \{G \in \mathcal{E}_{\rm bdry} : (E, G) \in \alpha\}$.

Proof. Let $E \in \mathcal{E}_{\rm bdry}$ be such that $\gamma_1(E) \neq \gamma_2(E)$. Let $D, F \in \mathcal{E}_{\rm bdry}$, $A, B \in \mathcal{E} \sim \mathcal{E}_{\rm bdry}$ and $T, U \in \mathcal{T}_{\rm bdry}$ be such that $T \neq U$, $\mathbf{e}(T) = \{D, E, A\}$ and $\mathbf{e}(U) = \{E, F, B\}$. We need to show that

$$\gamma_1(D) = \gamma_2(D) \quad \text{and} \quad \gamma_1(F) = \gamma_2(F). \tag{1}$$

Then *either* (I) $\mathbf{v}(D) \cap \mathbf{v}(E) \cap \mathbf{v}(F) = \emptyset$ *or* (II) $\mathbf{v}(D) \cap \mathbf{v}(E) \cap \mathbf{v}(F) \neq \emptyset$.

Case One. Suppose (I) holds.

Let a, b, c, d be the common vertices of D and A, D and E, E and F, F and B, respectively. By Lemma 4.3 we have $\{\gamma_1(D), \gamma_2 D\} \subset \{a, b\}$ and $\{\gamma_1(F), \gamma_2 F\} \subset \{c, d\}$.

Subcase One. $\gamma_i(E) \notin \mathbf{v}(E)$ for some $i \in \{1, 2\}$.

Let j be such that $\{i, j\} = \{1, 2\}$. Then $\gamma_i(D) = b$ and $\gamma_i(F) = c$ by Lemma 4.3. Now $\mathbf{p}(\gamma_i) \cap E = E$ so that $\mathbf{p}(\gamma_j) \cap E = E$. If $\gamma_j(E) \notin \mathbf{v}(E)$ then $\gamma_j(D) = b$ and $\gamma_j(F) = c$ so (1) holds. So suppose $\gamma_2(E)$

Subcase Two. $\{i, j\} = \{1, 2\}$, $\gamma_i(E) = b$ and $\gamma_j(E) = c$.

We cannot have $\gamma_i(F) = d$ by Lemma 4.4 nor can we have $\gamma_i(F) \in F \sim \mathbf{v}(F)$ by Lemmas 4.3 and 4.2. So $\gamma_i(F) = c$. We cannot have $\gamma_j(D) = a$ by Lemma 4.4 nor can we have $\gamma_j(D) \in D \sim \mathbf{v}(D)$ by Lemmas 4.3 and 4.2. So $\gamma_j(D) = b$. Since $\gamma_i(F) = c$ we cannot have $\gamma_i(D) = a$ by Lemma 4.4 nor can we have $\gamma_i(D) \in D \sim \mathbf{v}(D)$ by Lemmas 4.3 and 4.2. So $\gamma_i(D) = b$. Since $\gamma_j(D) = b$ we cannot have $\gamma_j(F) = d$ by Lemma 4.4 nor can we have $\gamma_j(F) \in F \sim \mathbf{v}(F)$ by Lemmas 4.3 and 4.2. So $\gamma_j(F) = c$.

Case Two. Suppose (II) holds. Let a be the common vertex of D, E, F and let b, c, d be such that $\mathbf{v}(D) = \{a, b\}$, $\mathbf{v}(E) = \{a, c\}$, $\mathbf{v}(F) = \{a, d\}$.

Subcase One. $\{i, j\} = \{1, 2\}$ and $\gamma_i(E) \notin \mathbf{v}(E)$.

Then $\gamma_i(D) = a$ and $\gamma_i(F) = c$ so $\mathbf{p}(\gamma_i) \cap \mathbf{int}\,(T \cup U) = \mathbf{c}(S) \sim \mathbf{v}(S)$ where $S = \mathbf{c}(a, \gamma_i(E))$. This implies $\mathbf{p}(\gamma_j) \cap \mathbf{int}\,(T \cup U) = \mathbf{c}(S) \sim \mathbf{v}(S)$ so that $\gamma_2(E) = \gamma_1(E)$ which contradicts our hypothesis. So this Subcase does not occur.

Subcase Two. $\{i, j\} = \{1, 2\}$, $\gamma_i(E) = a$ and $\gamma_j(E) = c$. We cannot have $\gamma_j(D) = b$ by Lemma 4.4 nor can we have $\gamma_j(D) \in D \sim \mathbf{v}(D)$ by Lemmas 4.3 and 4.2. So $\gamma_j(D) = a$. We cannot have $\gamma_j(F) = d$ by Lemma 4.3 nor can we have $\gamma_j(F) \in D \sim \mathbf{v}(F)$ by Lemmas 4.3 and 4.2. So $\gamma_j(F) = a$. Keeping in mind Lemma 4.1 this implies $\mathbf{p}(\gamma_j) \cap \mathbf{int}\,(T \cup U) = E \sim \mathbf{v}(E)$. But as $\gamma_i(D) \in D$ and $\gamma_i(F) \in F$ we find that $\mathbf{p}(\gamma_i) \cap \mathbf{int}\, T \cup U = \emptyset$. Thus this Subcase does not occur. □

4.4. **A useful classification.** Suppose $\gamma \in \Gamma$.

For each $v \in \mathcal{V}$ we let

$$\mathbf{u}_{\rm sub}(\gamma, v) = \begin{cases} \{v\} & \text{if sub = in and } v \in \mathcal{V}_{\rm in} \sim \mathbf{p}(\gamma); \\ \{v\} & \text{if sub = out and } v \in \mathcal{V}_{\rm out} \sim \mathbf{p}(\gamma); \\ \{v\} & \text{if } v \in \mathbf{p}(\gamma). \end{cases}$$

For each $E \in \mathcal{E}$ we let

$$\mathbf{u}_{\rm sub}(\gamma, E) = \begin{cases} E \sim \mathbf{v}(E) & \text{if sub = in and } \mathbf{v}(E) \subset \mathcal{V}_{\rm in}; \\ E \sim \mathbf{v}(E) & \text{if sub = out and } \mathbf{v}(E) \subset \mathcal{V}_{\rm out}; \\ \{v\} & \text{if sub = out and } v \in \mathcal{V}_{\rm out} \sim \mathbf{p}(\gamma); \\ \{v\} & \text{if } v \in \mathbf{p}(\gamma). \end{cases}$$

Keeping in mind that

$$E \cap \mathbf{p}(\gamma) \in \mathbf{E} \cup \mathbf{V}$$

we find that there are unique functions

$$\mathbf{u}_{\rm in}, \mathbf{u}_{\rm out} : \mathcal{E}_{\rm bdry} \to [0,1]$$

such that $\mathbf{u}_{\rm in} \le \mathbf{u}_{\rm out}$ and

$$E \cap \mathbf{p}(\gamma) = \{(1-t)\mathbf{v}_{\rm in}(E) + t\mathbf{v}_{\rm out}(E) : \mathbf{u}_{\rm in}(E) \le t \le \mathbf{u}_{\rm out}(E)\}.$$

We let

$$\mathbf{U}_{\rm in}$$

be the union of the set

$$\{v \in \mathcal{V}_{\rm in} : v \notin \mathbf{p}(\gamma)\};$$

the sets

$$E \sim \mathbf{v}(E) \quad \text{corresponding to } E \in \mathcal{E}_{\rm in};$$

the sets

$$\{(1-t)\mathbf{v}_{\rm in}(E) + t\mathbf{v}_{\rm out}(E) : 0 < t < \mathbf{u}_{\rm in}(E)]\} \quad \text{corresponding to } E \in \mathcal{E}_{\rm bdry};$$

the sets

$$\mathbf{h}_+(\gamma(E), \gamma(F)) \cap \mathbf{int}\, T$$

corresponding to $T \in \mathcal{T}_{\rm bdry}$ and E, F such that $\mathcal{E}_{\rm bdry} \cap \mathbf{e}(T) = \{E, F\}$, $F = \sigma[E]$ and $\gamma(E) \ne \gamma(F)$; the sets

$$\mathbf{int}\, T$$

corresponding to $T \in \mathcal{T}_{\rm bdry}$ and E, F such that $\mathcal{E}_{\rm bdry} \cap \mathbf{e}(T) = \{E, F\}$, $F = \sigma[E]$ and $\gamma(E) = \gamma(F) \in \mathcal{V}_{\rm out}$; and the sets

$$\mathbf{int}\, T \quad \text{corresponding to } T \in \mathcal{T}_{\rm in}.$$

We let

$$\mathbf{U}_{\rm out}$$

be the union of the set

$$\{v \in \mathcal{V}_{\rm out} : v \notin \mathbf{p}(\gamma)\};$$

the sets

$$E \sim \mathbf{v}(E) \quad \text{corresponding to } E \in \mathcal{E}_{\rm out};$$

the sets

$$\{(1-t)\mathbf{v}_{\rm in}(E) + t\mathbf{v}_{\rm out}(E) : \mathbf{u}_{\rm out}(E) < t < 1\} \quad \text{corresponding to } E \in \mathcal{E}_{\rm bdry};$$

the sets

$$\mathbf{h}_+(\gamma(E), \gamma(F)) \cap \mathbf{int}\, T$$

corresponding to $T \in \mathcal{T}_{\rm bdry}$ and E, F such that $\mathcal{E}_{\rm bdry} \cap \mathbf{e}(T) = \{E, F\}$, $F = \sigma[E]$ and $\gamma(E) \ne \gamma(F)$; the sets

$$\mathbf{int}\, T$$

corresponding to $T \in \mathcal{T}_{\rm bdry}$ and E, F such that $\mathcal{E}_{\rm bdry} \cap \mathbf{e}(T) = \{E, F\}$, $F = \sigma[E]$ and $\gamma(E) = \gamma(F) \in \mathcal{V}_{\rm in}$; and the sets

$$\mathbf{int}\, T \quad \text{corresponding to } T \in \mathcal{T}_{\rm out}.$$

Proposition 4.11. The sets $\mathbf{U}_{\rm in}$ and $\mathbf{U}_{\rm out}$ are open. $\mathbb{R}^2$ is the disjoint union of $\mathbf{U}_{\rm in}$, $\mathbf{U}_{\rm out}$ and $\mathbf{p}(\gamma)$. We have

$$\mathcal{V}_{\rm in} \sim \mathbf{p}(\gamma) \subset \mathbf{U}_{\rm in} \quad \text{and} \quad \mathcal{V}_{\rm out} \sim \mathbf{p}(\gamma) \subset \mathbf{U}_{\rm out}.$$

4.5. **Types of edges with respect to $\gamma \in \Gamma$.**

(i) $\gamma(E) \notin \mathbf{v}(E)$ and there is one and only one $F \in \mathcal{C}(-1,1)$ such that $E = F_0$,

$$[\gamma(F_{-1}),\gamma(F_0),\gamma(F_1)] = 0, \quad [\mathbf{v}_{\rm in}(F_0),\mathbf{v}_{\rm out}(F_0),\gamma(F_{-1})] < 0, \quad [\mathbf{v}_{\rm in}(F_0),\mathbf{v}_{\rm out}(F_0),\gamma(F_1)] > 0.$$

(ii) $\gamma(E) \notin \mathbf{v}(E)$ and there is one and only one $F \in \mathcal{C}(-1,1)$ such that $E = F_0$,

$$[\gamma(F_{-1}),\gamma(F_0),\gamma(F_1)] \neq 0, \quad [\mathbf{v}_{\rm in}(F_0),\mathbf{v}_{\rm out}(F_0),\gamma(F_{-1})] < 0, \quad [\mathbf{v}_{\rm in}(F_0),\mathbf{v}_{\rm out}(F_0),\gamma(F_1)] > 0.$$

(iii) $\gamma(E) \notin \mathbf{v}(E)$ and there is one and only one $F \in \mathcal{C}(-1,1)$ such that $E = F_0$, $\gamma(F_{-1}) = \mathbf{v}_{\rm in}(F_0)$ and $\gamma(F_1) = \mathbf{v}_{\rm out}(F_0)$.

(iv) $\gamma(E) \notin \mathbf{v}(E)$ and there is one and only one $F \in \mathcal{C}(-1,1)$ such that $E = F_0$, $\gamma(F_{-1}) \in \mathbf{v}(E)$ and

$$[\gamma(F_{-1}),\gamma(F_0),\gamma(F_1)] \neq 0.$$

(v) $\gamma(E) \notin \mathbf{v}(E)$ and there are exactly two $F \in \mathcal{C}(-1,1)$ such that $E = F_0$ and $\{\gamma(F_{-1},\gamma(F_1)\} = \mathbf{v}(F_0)$.

(vi) $\gamma(E) \in \mathbf{v}(E)$ and $\gamma(F) = \gamma(E)$ for all $F \in \mathbf{o}(E)$;

(vii) $\gamma(E) \in \mathbf{v}(E)$ and there is exactly one $F \in \mathcal{C}(-1,1)$ such that $E = F_0$, $\mathbf{v}_{\rm in}(F_0) = \gamma(F_{-1})$ and $\mathbf{v}_{\rm out}(F_0) = \gamma(F_1)$.

(viii) $\gamma(E) \in \mathbf{v}(E)$ and there are $I, J \in \mathbb{Z}$ and $F \in \mathcal{C}(I,J)$ such that $I < 0 < J$, $E = F_0$, $\gamma(F_i) = \gamma(F_0)$ whenever $i \in \mathbb{I}(I+1, J-1)$, there is $t \in (0,1)$ such that $\gamma(F_0) = (1-t)\gamma(F_I) + t\gamma(F_J)$, there is $H \in \mathbf{H}$ such that $\gamma(F_0) \in$ **bdry** H and $\cup_{i=I}^{J} F_i \subset H$;

(ix) $\gamma(E) \in \mathbf{v}(E)$ and there are $I, J \in \mathbb{Z}$ and $F \in \mathcal{C}(I,J)$ such that $I < 0 < J$ $E = F_0$, $\gamma(F_i) = \gamma(F_0)$ whenever $i \in \mathbb{I}(I+1, J-1)$, $[\gamma(F_I),\gamma(F_0),\gamma(F_J)] \neq 0$ and such that

$$\left(\cup_{i=I+1}^{J-1} F_i\right) \cap \{\gamma(F_0) + s((1-t)\gamma(\gamma(F_I) + t\gamma(F_J)) : 0 < s < \infty \text{ and } 0 \le t \le 1\} = \emptyset.$$

(x) $\gamma(E) \in \mathbf{v}(E)$ and there are $I, J \in \mathbb{Z}$ and $F \in \mathcal{C}(I,J)$ such that $I < 0 < J$ $E = F_0$, $\gamma(F_i) = \gamma(F_0)$ whenever $i \in \mathbb{I}(I+1, J-1)$, $[\gamma(F_I),\gamma(F_0),\gamma(F_J)] \neq 0$ and such that

$$\left(\cup_{i=I+1}^{J-1} F_i\right) \sim \{\gamma(F_0)\} \subset \{\gamma(F_0) + s((1-t)\gamma(\gamma(F_I) + t\gamma(F_J)) : 0 < s < \infty \text{ and } 0 \le t \le 1\} = \emptyset.$$

4.6. **Types of edges with respect to $\gamma \in \Gamma$.** Suppose $E \in \mathcal{E}_{\rm bdry}$. Let $E_i = \sigma^i[E]$ and let $g_i = \gamma(E_i)$ for $i \in \mathbb{Z}$.

Exactly one of the following statements holds:

(I) $g_0 \notin \mathbf{v}(E_0)$;
(II) $g_0 \in \mathbf{v}(E_0)$ and $g_i = g_0$ for all $i \in \mathbb{Z}$;
(III) $g_0 \in \mathbf{v}(E_0)$ and there is one and only one $(I,J) \in \mathbb{Z}^2$ such that $I < 0 < J$; $g_i = g_0$ if $i \in \mathbb{I}(I+1, J-1)$ and $g_0 \notin \{g_I, g_J\}$.

If (I) holds then exactly one of the following statements holds:

(i) $\{g_{-1}, g_1\} \cap E_0 = \emptyset$ and $\{g_{-1}, g_0, g_1\}$ is nonlinear;
(ii) $\{g_{-1}, g_1\} \cap E_0 = \emptyset$ and $\{g_{-1}, g_0, g_1\}$ is linear;
(iii) $\{g_{-1}, g_1\} \cap \mathbf{v}(E) \neq \emptyset$ and $\{g_{-1}, g_0, g_1\}$ is nonlinear;
(iv) $g_{-1} = g_1$;
(v) $\{g_{-1}, g_1\} = \mathbf{v}(E)$.

If g_0, I, J are as in (III) holds then exactly one of the following statements holds:

(vii) $g_0 \in \mathbf{v}(E)$ and $g_{-1} = g_0$;

(viii) $g_0 \in \mathbf{v}(E)$ and there are $I, J \in \mathbb{Z}$ such that $I < 0 < J$, $g_i = g_0$ whenever $i \in \mathbb{I}(I+1, J-1)$, there is $t \in (0,1)$ such that $g_0 = (1-t)g_I + tg_J$, there is $H \in \mathbf{H}$ such that $\gamma(F_0) \in \mathbf{bdry}\, H$ and $\cup_{i=I}^{J} F_i \subset H$;

(ix) $g_0 \in \mathbf{v}(E)$ and there are $I, J \in \mathbb{Z}$ such that $I < 0 < J$ $g_i = g_0$ whenever $i \in \mathbb{I}(I+1, J-1)$, $\{g_I, g_0, g_J\}$ is nonlinear and such that

$$\left(\cup_{i=I+1}^{J-1} E_i\right) \cap \{g_0 + s((1-t)g_I + tg_J) : 0 < s < \infty \text{ and } 0 \leq t \leq 1\} = \emptyset.$$

(x) $g_0 \in \mathbf{v}(E)$ and there are $I, J \in \mathbb{Z}$ such that $I < 0 < J$ $g_i = g_0$ whenever $i \in \mathbb{I}(I+1, J-1)$, $\{g_I, g_0, g_J\}$ is nonlinear and such that

$$\left(\cup_{i=I+1}^{J-1} F_i\right) \sim \{g_0\} \subset \{g_0 + s((1-t)(g_I + tg_J) : 0 < s < \infty \text{ and } 0 \leq t \leq 1\} = \emptyset.$$

4.7. **Minimizers .**

Definition 4.11. We let

$$\Gamma_{\min} = \{\gamma \in \Gamma : \mathbf{l}(\gamma) \leq \mathbf{l}(\delta) \text{ whenever } \delta \in \Gamma\}.$$

The members of $\Gamma_{\min}$ are called **minimizers**.

Proposition 4.12. $\Gamma_{\min}$ is nonempty.

Proof. Let

$$F : [0,1]^{\mathcal{E}_{\text{bdry}}} \to \Gamma \quad \text{and} \quad L : [0,1]^{\mathcal{E}_{\text{bdry}}} \to [0, \infty)$$

be such that

$$F(c)(E) = (1-c(\gamma))\mathbf{v}_{\text{in}}(E) + c(\gamma)\mathbf{v}_{\text{out}}(E) \quad \text{and} \quad L(c) = \mathbf{l}(F(c)) \quad \text{for } c \in [0,1]^{\mathcal{E}_{\text{bdry}}};$$

then F is univalent with range Γ and L is convex on the compact cube $[0,1]^{\mathcal{E}_{\text{bdry}}}$. □

4.8. **Special paths.** A number of properties of a member of $\Gamma_{\min}$ are affinely invariant; these properties are used to define the class Γ_{special}.

Definition 4.12. Suppose $I \in \mathbb{Z}$ and $n \in \mathbb{N}^+$. We let

$$\mathcal{C}(I, n)$$

be the set of maps $E : \mathbb{I}(I, n) \to \mathcal{E}_{\text{bdry}}$ such that

(i) $(E_i, E_{i+1}) \in \alpha$ whenever $i \in \mathbb{I}(I, J)$ and $i + 1 \leq J$;

(ii) $E_i \neq E_{i+2}$ whenever $i \in \mathbb{I}(I, J)$ and $i + 2 \leq J$.

Proposition 4.13. Suppose $I, J \in \mathbb{Z}$ and $E : \mathbb{I}(I, J) \to \mathcal{E}_{\text{bdry}}$. Then $E \in \mathcal{C}(I, J)$ if and only if *either* $E_i = \sigma^i[E_I]$ for $i \in \mathbb{I}(I, J)$ *or* $E_i = \sigma^{J-i}[E_J]$ for $i \in \mathbb{I}(I, J)$.

Proof. □

Definition 4.13. Suppose $\gamma \in \Gamma$. We say γ is **special** if the following three conditions hold:

(I) The points

$$\gamma(E_1), \gamma(E_2), \gamma(E_3)$$

are distinct and collinear whenever $E \in \mathcal{C}(1, 3)$ and $\gamma(E_2) \notin \mathbf{v}(E_2)$.

(II) We have

$$\gamma(E_2) = v$$

whenever $E \in \mathcal{C}(1, 3)$, $v \in \mathbf{V}$, $\mathbf{v}(E_1) \cap \mathbf{v}(E_2) \cap \mathbf{v}(E_3) = \{v\}$, $\gamma(E_1) = v$ and $\gamma(E_3) = v$.

(III) We have
$$\gamma(E_i) \neq v \quad \text{for } i \in \mathbb{I}(I+1, J-1)$$
whenever $I, J \in \mathbb{Z}$, $I+1 \leq J-1$, $E \in \mathcal{C}(I,J)$ $v \in \mathbf{V}$ and $G, H \in \mathbf{H}$ are such that
(i) the points $\gamma(E_I), v, \gamma(E_J)$ are distinct;
(ii) $\mathbf{c}(\gamma(E_I), v) \subset \mathbf{bdry}\, G$ and $\gamma(E_J) \in \mathbf{int}\, H$;
(iii) $\mathbf{c}(\gamma(E_J), v) \subset \mathbf{bdry}\, H$ and $\gamma(E_I) \in \mathbf{int}\, H$;
(iv) $v \in \mathbf{v}(E_i)$ and $E_i \subset H \cap I$ whenever $i \in \mathbb{I}(()I, J)$ and $I < i < J$.

We let
$$\Gamma_{\text{special}} = \{\gamma \in \Gamma : \gamma \text{ is special}\}.$$

Remark 4.1. By a straightforward argument we shall give in Lemma 4.5 we will show that
$$\Gamma_{\min} \subset \Gamma_{\text{special}}.$$

Suppose $\gamma \in \Gamma$ and
$$\iota : \mathbf{O} \to \mathcal{E}_{\text{bdry}} \times \mathcal{V}$$
is such that $\gamma(E) = v$ whenever $\mathcal{O} \in \mathbf{O}$ and $(E, v) = \iota(\mathcal{O}).\gamma(E) = v$. In 6 we will give an algorithm which computes γ up to equivalence, given α and ι in time $O(N^2)$ where N is the cardinality of $\mathcal{E}_{\text{bdry}}$. We will show that if $\mathcal{V} = A[\mathbb{Z}^2$ for some affine isomorphism of $\mathbb{R}^2$ that this algorithm runs in time $O(N)$ given α; moreover, in this case, the algorithm uses only integer arithmetic.

We will show in 5 that for each $\mathcal{O} \in \mathbf{O}$ there is a nonempty set of pairs (v, E) such that $E \in \mathcal{O}$, $v \in \mathbf{v}(E)$ and such that $\gamma(E) = v$ whenever $\gamma \in \Gamma_{\text{special}}$. *This implies that* Γ_{special} *has, up to equivalence, a unique member and that* $\Gamma_{\min} = \Gamma_{\text{special}}$.

In what follows we will need to make use of the following Proposition.

Proposition 4.14. Suppose $E \in \mathcal{E}_{\text{bdry}}$, $I \in \mathbb{N}^+$, $a \in \cap\{\mathbf{v}(\sigma^i[E]) : i \in \mathbb{I}(0, I)\}$, $\gamma \in \Gamma_{\text{special}}$ and $\gamma(E) = a$. Then $\gamma(\sim iE) = a$ whenever $i \in \mathbb{I}(1, I-1)$.

Proof. Were the Proposition false there would be $j \in \mathbb{I}(1, I-1)$ such that $\gamma(\sigma^i[E]) = a$ for $j \in \mathbb{I}(0, j-1)$ but such that $\gamma(\sigma^j[E]) \neq a$. Suppose $\gamma(\sigma^j[E]) \in \mathbf{v}(\sigma^j[E])$. Since $\gamma(\sigma^j[E]) \neq a$, this is incompatible with (III) of Definition 4.13. Suppose $\gamma(\sigma^j[E] \notin \mathbf{v}(\sigma^j[E])$. Since $\gamma(\sigma^{j-1}[E]) = a \in \mathbf{v}(\sigma^j[E])$ and since $\sigma^j[E] \cap \sigma^{j+1}[E] = \{a\}$ this is incompatible with (I) of Definition 4.13. □

4.9. **The affine invariance of** Γ_{special}**.** Suppose $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine isomorphism.

Let
$$\mathcal{T}_A = \{A[T] : T \in \mathcal{T}\}.$$
Evidently, $\mathcal{T}_A$ is a triangulation of $\mathbb{R}^2$. and let $\mathcal{E}_A$ and $\mathcal{V}_A$ be the corresponding sets of edges and vertices.

Let
$$\mathcal{V}_{\text{in},A} = A[\mathcal{V}_{\text{in}}] \quad \text{and let} \quad \mathcal{V}_{\text{out},A} = A[\mathcal{V}_{\text{out}}]$$
and let
$$\mathcal{E}_{\text{in,A}}, \quad \mathcal{E}_{\text{out,A}}, \quad \mathcal{E}_{\text{bdry,A}}, \quad \mathcal{T}_{\text{in,A}}, \quad \mathcal{T}_{\text{out,A}}, \quad \mathcal{T}_{\text{bdry,A}}, \quad \mathbf{v}_{\text{in,A}}, \quad \mathbf{v}_{\text{out,A}}$$
be as in 4 with $\mathcal{V}_{\text{in}}$ and $\mathcal{V}_{\text{out}}$ there replaced by $\mathcal{V}_{\text{in},A}$ and $\mathcal{V}_{\text{out},A}$, respectively.

Let
$$\alpha_A$$

be as in Definition 4.1 with $\mathcal{T}_{\rm bdry}$ there replaced by $\mathcal{T}_{{\rm bdry},A}$.

$$\alpha_A = \{(A[E], A[F]) : (E,F) \in \alpha\}.$$

Suppose $I, J \in \mathbb{Z}$. We let

$$\mathcal{C}_A(I,J)$$

be defined as in Definition 4.12 with $\mathcal{E}_{\rm bdry}$ and α there replaced by $\mathcal{E}_{{\rm bdry},A}$ and α_A, respectively. Evidently,

$$\mathcal{C}_A(I,J) = \{E_A : E \in \mathcal{C}(I,J)\}$$

where $E_A : \mathbb{I}(I,J) \to \mathcal{E}_{{\rm bdry},A}$ is such that $(E_A)_i = A[E_i]$ whenever $E : \mathbb{I}(I,J) \to \mathcal{E}_{\rm bdry}$.

Let

$$\Gamma_A$$

be the set of choice functions for $\mathcal{E}_{{\rm bdry},A}$; let

$$\mathbf{p}_A, \quad \mathbf{l}_A$$

be as in Definition 4.8 with Γ and $\mathcal{E}_{\rm bdry}$ there replaced by Γ_A and $\mathcal{E}_{{\rm bdry},A}$, respectively; let

$$\Gamma_{{\rm min},A}$$

be as in Definition 4.11 with $\mathcal{E}_{\rm bdry}$ there replaced by $\mathcal{E}_{{\rm bdry},A}$, respectively, and let

$$\Gamma_{{\rm special},A}$$

be as in Definition 4.13 with $\mathcal{E}_{\rm bdry}$ and $\mathcal{C}(I,J)$ there replaced by $\mathcal{E}_{{\rm bdry},A}$ and $\mathcal{C}_A(I,J)$, respectively. For each $\gamma \in \Gamma$ let $\gamma_A \in \Gamma_A$ be such that

$$\gamma_A(A[E]) = A(\gamma(E)) \quad \text{for } E \in \mathcal{E}_{\rm bdry}.$$

Then

$$\mathbf{p}_A(\gamma_A) = A[\mathbf{p}(\gamma)] \quad \text{for } \gamma \in \Gamma$$

and

$$\Gamma_{{\rm special},A} = \{\gamma_A : \gamma \in \Gamma_{\rm special}\}.$$

Now

$$\mathbf{l}_A(\gamma_A) \neq \mathbf{l}(\gamma)$$

generically in A and γ; nonetheless, because $\Gamma_{\rm min} = \Gamma_{\rm special}$ we have

$$\Gamma_{{\rm min},A} = \{\gamma_A : \gamma \in \Gamma_{\rm min}\}.$$

Lemma 4.5. $\Gamma_{\rm min} \subset \Gamma_{\rm special}$.

Proof. Suppose $\gamma \in \Gamma_{\rm min}$. It is obvious that (I) and (II) of Definition 4.13 hold.

Suppose I, J, E, v are as in (III) (i)-(iv) of Definition 4.13 but that $\gamma(E_i) = v$ for some $i \in \mathbb{I}(I+1, J+1)$. Let $I^*, J* \in \mathbb{I}(I,J)$ be such that $I^* < i < J*$, $\gamma(E_{I*}) \neq v$, $\gamma(E_{J*}) \neq v$ and $\gamma(E_j) = v$ if $j \in \mathbb{I}(()I^*+1, J^*-1)$. For each $i \in \mathbb{I}(I^*+1, J^*-1)$ let w_i be such that $\mathbf{v}(E_i) = \{v, w_i\}$. Let L be a line such that

(i) v and $\gamma(E_{I^*})$ lie on different sides of L;
(ii) v and $\gamma(E_{J^*})$ lie on different sides of L;
(iii) for each $i \in \mathbb{I}(I^*+1, J^*-1)$, v and w_i.

For each $i \in \mathbb{I}(I^*+1, J^*-1)$ let x_i be such that $L \cap E_i = \{x_i\}$. Let $\delta \in \Gamma$ be such that $\delta(F) = \gamma(F)$ when $F \neq E_i$ for $i \in \mathbb{I}(I^*+1, J^*-1)$ and $\delta(E_i) = x_i$ when $i \in \mathbb{I}(I^*+1, J^*-1)$. Then $\mathbf{l}(\delta) < \mathbf{l}(\gamma)$ which contradicts the minimality of γ. Thus (III) of Definition 4.13 holds. □

Proposition 4.15. Suppose

(i) $I, J \in \mathbb{Z}$, $I < J$ and $E \in \mathcal{C}(I, J)$;
(ii) $\omega \in \mathbb{R}_2 \sim \{0\}$ and $m \in \mathbb{R}$;
(iii) for each $i \in \mathbb{I}(I, J)$,

$$E_i \cap \{\omega \leq m\} \neq \emptyset \quad \text{and} \quad E_i \cap \{\omega > m\} \neq \emptyset;$$

(iv) $M = \min\{\max\{\omega(x) : x \in E_i\} : i \in \mathbb{I}(I, J)\}$.

Then the following statements hold:

(v) $m < M < \infty$.
(vi) For each $i \in \mathbb{I}(I, J)$ there are $d_i, e_i \in \mathbf{V}$ such that $E_i \cap \{\omega = m\} = \{d_i\}$ and $E_i \cap \{\omega = M\} = \{e_i\}$.
(vii) We have

$$\mathbf{j}_-(E_i) \cap \{m < \omega < M\} \subset \mathbf{int}\, \mathbf{j}_-(E_j) \quad \text{whenever } i, j \in \mathbb{I}(I, J) \text{ and } i < j.$$

(viii) If $a \in \mathbf{j}_-(E_I) \cap \{\omega = m\}$, $b \in \mathbf{j}_+(E_J) \cap \{\omega = m\}$, $a \neq b$, and, for each $i \in \mathbb{I}(I, J)$, $t_i \in \mathbb{R}$ is such that $d_i = (1 - t_i) + t_i b$ then $t_i \leq t_j$ whenever $i, j \in \mathbb{I}(I, J)$ and $i < j$ with equality only if $d_i = d_j$.

Proof. In view of (iii), for any $y \in \mathbb{R}$ and $i \in \mathbb{I}(I, J)$ the line containing E_i meets $\{\omega = y\}$ transversely at a point $f_i(y)$; in particular, (v) and (vi) hold and $\mathbf{v}(E_i) \cap \{m < \omega < M\} = \emptyset$ whenever $i \in \mathbb{I}(I, J)$. It follows that $f_i(y) \neq f_j(y)$ whenever $y \in (m, M)$, $i, j \in \mathbb{N}I, J$ and $i < j$ since two distinct members of $\mathcal{E}_{\text{bdry}}$ can only meet in a vertex. For each $y \in (m, M)$ and $i \in \mathbb{I}(I, J)$ let $u_i(y) \in \mathbb{R}$ be such that $f_i(y) = (1 - u_i(y)) f_I(y) + u_i(y) f_J(y)$. In particular, $E_i \neq E_j$ if $i, j \in \mathbb{I}(I, J)$ and $i < j$.

Suppose $y \in (m, M)$. If $i \in \mathbb{I}(I, J)$ and $I < i < J$ then *either* (a) $u_{i-1}(y) < u_i(y) < u_{i+1}(y)$ *or* (b) $u_{i-1}(y) > u_i(y) > u_{i+1}(y)$ since $f_{i-1}(y) \in \mathbf{j}_-(E_i)$ and $f_{i+1}(y) \in \mathbf{j}_+(E_i)$. It follows that *either* (c) $u_i(y) < u_{i+1}(y)$ whenever $i \in \mathbb{I}(I, J)$ and $i < I$ *or* (d) $u_i(y) > u_{i+1}(y)$ whenever $i \in \mathbb{I}(I, J)$ and $i < I$. Since $u_I(y) = 0$ and $u_J(y) = 1$ we find that (d) holds. Thus

$$u_i(y) < u_j(y) \quad \text{whenever } i, j \in \mathbb{I}(I, J) \text{ and } i < j.$$

Thus (vii) holds. (viii) follows easily from (vii). □

Theorem 4.2. Suppose

(i) $E \in \mathcal{E}_{\text{bdry}}$ and $E_i = \sigma^i[E]$ for $i \in \mathbb{Z}$;
(ii) $I, J \in \mathbb{Z}$ and $I \leq J$;
(iii) $\omega \in \mathbb{R}_2 \sim \{0\}$ and $m \in \mathbb{R}$;
(iv) for each $i \in \mathcal{I}$,

$$E_i \cap \{\omega \leq m\} \neq \emptyset \quad \text{and} \quad E_i \cap \{\omega > m\} \neq \emptyset;$$

(v) $\gamma \in \Gamma_{\text{special}}$;
(vi) $a \in \mathbf{c}(\gamma(E_{I-1}), \gamma(E_I))$, $b \in \mathbf{c}(\gamma(E_J), \gamma(E_{J+1}))$, $a \neq b$ and $\{a, b\} \subset \{\omega = m\}$.

Then

$$\omega(\gamma(E_i)) = m \quad \text{for } i \in \mathbb{I}(I - 1, J + 1).$$

Proof. Applying a translation if necessary, we may assume without loss of generality that $m = 0$. Let $L = \{\omega = 0\}$. For each $i \in \mathbb{Z}$ let $g_i = \gamma(E_i)$.

I claim that

$$\omega(g_i) = 0 \quad \text{for } i \in \mathbb{I}(I,J). \tag{2}$$

So suppose (2) does not hold. Then $N = \max\{|\omega(g_i)| : i \in \mathbb{I}(I,J)\} > 0$ and there will exist $I^* \in \mathbb{I}(I,J)$ such that

$$|\omega(g_{I^*})| = N \quad \text{and} \quad -N < \omega(g_i) < N \quad \text{if } i \in \mathbb{I}(I,J) \text{ and } i < I^*. \tag{3}$$

Lemma 4.6. $g_{I^*} \in \mathbf{v}(E_{I^*})$.

Proof. Suppose, contrary to the Lemma, $g_{I^*} \notin \mathbf{v}(E_{I^*})$. Since γ is special the points $g_{I^*-1}, g_{I^*}, g_{I^*+1}$ would be distinct and collinear. This is impossible if $I < I^* < J$ in view of (3).

Suppose $I = I^* < J$. Then $a = (1-s)g_{I^*-1} + sg_{I^*}$ for some $s \in [0,1]$ so $0 = (1-s)\omega(g_{I^*-1}) + s\omega(g_{I^*})$. Since $|\omega(g_{I^*+1})| \le N$ the points $g_{I^*-1}, g_I^*, g_{I^*+1}$ cannot be collinear. By a similar argument one arrives at a contradiction if $I < I^* = J$.

So suppose $I = I^* = J$. Then $0 = (1-s)\omega(g_{I^*-1}) + s\omega(g_{I*})$ for some $s \in [0,1]$ and $0 = (1-t)\omega(g_I) + t\omega(g_{I*+1})$ for some $t \in [0,1]$ so the points $g_{I^*-1}, g_I^*, g_{I^*+1}$ cannot be collinear. □

Let $d_{I-1} = a$ and let $d_{J+1} = b$. Since $E_{I-1} \cup E_I \subset \mathbf{j}_-(E_I)$ and $\mathbf{j}_-(E_I)$ is convex we find that $d_{I-1} \in \mathbf{j}_-(E_I)$. Since $E_J \cup E_{J+1} \subset \mathbf{j}_+(E_J)$ and $\mathbf{j}_+(E_J)$ is convex we find that $d_{J+1} \in \mathbf{j}_+(E_J)$. Applying Proposition 4.15 we obtain for each $i \in \mathbb{I}(I,J)$ a number $t_i \in [0,1]$ such that if $d_i = (1-t_i)a + t_ib$ then then

$$E_i \cap L = \{d_i\} \tag{4}$$

and such that

$$t_i \le t_j \quad \text{and} \quad d_i \subset \mathbf{j}_-(E_j) \quad \text{whenever } i,j \in \mathbb{I}(I,J) \text{ and } i < j. \tag{5}$$

Lemma 4.7. There is one and only $s \in (-\infty, t_{I^*}]$ such that if $f_{I^*-1} = (1-s)a + sb$ then g_{I^*-1} lies on the line containing g_{I^*} and f_{I^*-1}.

Proof. In case $I^* = I$ we can take $s = 0$ so suppose $I^* > I$. Since $|\omega(g_{I^*-1})| < N = |\omega(g_{I^*})|$, $g_{I^*-1} \neq g_{I^*}$ and the line containing g_{I^*-1} and g_{I^*} meets L in a unique point f_{I^*-1}. Let $s \in \mathbb{R}$ be such that $f_{I^*-1} = (1-s)a + sb$. Then $f_{I^*-1} \in \mathbf{j}_-(E_{I^*})$ since $g_{I^*} \in E_{I^*} \subset \mathbf{j}_-(E_{I^*})$ and $g_{I^*-1} \in E_{I^*-1} \subset \mathbf{j}_-(E_{I^*})$. Since $L \cap \mathbf{j}_-(E_{I^*}) = \{(1-u)a + ub : u \in (-\infty, t_{I^*}]\}$ by virtue of Proposition 4.15 (viii) the Lemma is proved. □

Let $T \in \mathbf{H}$ be such that g_{I^*-1} and g_{I^*} belong to $\mathbf{bdry}\, T$ and $b \in T$. Then $e_j \in T$ whenever $I^* \le j$ so that $E_j \subset T$ whenever $I^* \le j$.

Next, let $\mathcal{J}$ be the set of $j \in \mathbb{I}(I,J)$ such that $I^* \le j$ and $g_j = g_{I^*}$ if $i \in \mathbb{I}(I,J)$ and $I^* \le i \le j$. Let $J^* = \max \mathcal{J}$. Since $g_{I^*} \in \mathbf{v}(E_{I^*})$ and two members of $\mathcal{E}_{\text{bdry}}$ can only meet in a common vertex we find that $g_j \in \mathbf{v}(E_j)$ if $j \in \mathcal{J}$.

Lemma 4.8. If $\omega(g_{J^*+1}) \neq \omega(g_{J^*})$ there is one and only $u \in [t_J^*, \infty)$ such that if $f_{J^*+1} = (1-u)a + ub$ then g_{J^*+1} lies on the line containing g_J^* and f_{J^*+1}.

Proof. Suppose $\omega(g_{J^*+1}) \neq \omega(g_J^*)$. If $J^* = J$ we may take $u = 1$ so suppose $J^* < J$. Since $\omega(g_{J^*+1}) < \omega(g_J^*)$, $g_{J^*+1} \neq g_J^*$ and the line containing g_{J^*+1} and g_J^* meets L in a unique point f_{J^*+1}. Let $u \in \mathbb{R}$ be such that $f_{J^*+1} = (1-u)a + ub$. Then $f_{J^*+1} \in \mathbf{j}_+(E_J^*)$ since $g_{J^*} \in E_{J^*} \subset \mathbf{j}_+(E_{J^*})$ and $g_{J^*+1} \in E_{J^*+1} \subset \mathbf{j}_+(E_J^*)$. Since $L \cap \mathbf{j}_+(E_J^*) = \{(1-w)a + wb : w \in [t_J^*, \infty)$ and the Lemma is proved. □

Suppose $\omega(g_{J^*+1}) \neq \omega(g^*_J)$ and f_{J^*+1} is as in the preceding Lemma. Let $U \in \mathbf{H}$ be such that g^*_J and g_{J^*+1} belong to $\mathbf{bdry}\, U$ and $a \in U$. Since $t_i \leq t_{J*}$ whenever $i \in \mathcal{J}$ and $f_{J^*+1} \in \mathbf{j}_+(E_{J^*})$ we find that $E_i \subset U$ whenever $i \in \mathcal{J}$. Since γ is special we are now in a contradiction.

In case $\omega(g^*_J) = \omega(g_{J^*+1})$ we note that, by the definition of $\mathcal{J}$, $g_{J^*+1} \neq g_{J^*}$ and we let $U \in \mathbf{H}$ be such that $\mathbf{bdry}\, U = \{\omega = \omega(g_{J^*})\}$ and $L \subset U$. If $j \in \mathcal{J}$ then one of the vertices of E_j is g_{J^*} and E_j meets L so that $E_j \subset U$. Since γ is special we are now in a contradiction.

Thus (2) holds.

Suppose $i \in \mathbb{I}(I, J)$. Since $g_i \in E_i$ and $\{\omega = 0\} \cap E_i = \{e_i\}$ we find that $g_i = e_i$.

Let u_0 be such that $a = (1 - u_0)\gamma(E_0) + u_0\gamma(E_1)$. Applying ω to this equation we find that $0 = (1 - u_0)\gamma(E_0)$. Let u_{I+1} be such that $b = (1 - u_{I+1})\gamma(E_I) + u_{I+1}\gamma(E_{I+1})$. Applying ω to this equation we find that $0 = u_{I+1}\gamma(E_{I+1})$. Thus the final assertion of the Theorem holds.

□

5. Locating vertices on a special γ.

Theorem 5.1. Suppose

(i) $\mathcal{O} \in \mathbf{O}$ and $V = \cup\{\mathbf{v}(E) : E \in \mathcal{O}\}$;

(ii) $\omega : \mathbb{R}^2 \to \mathbb{R}$, ω is linear,

$$M_{\text{in}} = \max\{\omega(x) : x \in V \cap \mathcal{V}_{\text{in}}\} \quad \text{and} \quad M_{\text{out}} = \max\{\omega(x) : x \in V \cap \mathcal{V}_{\text{out}}\}.$$

Then

$$\textit{either} \quad M_{\text{in}} < M_{\text{out}} \quad \textit{or} \quad M_{\text{out}} < M_{\text{in}}.$$

Moreover, in case $M_{\text{in}} < M_{\text{out}}$, then

$$\mathcal{F} = \{F \in \mathcal{O} : F \subset \{\omega > M_{\text{in}}\} \text{ and } \omega(\mathbf{v}_{\text{in}}(F)) = M_{\text{in}}\} \neq \emptyset$$

and

$$\gamma(F) = \mathbf{v}_{\text{in}}(F) \quad \text{whenever } F \in \mathcal{F} \text{ and } \gamma \in \Gamma_{\text{special}}$$

and, in case $M_{\text{out}} < M_{\text{in}}$,

$$\mathcal{F} = \{F \in \mathcal{O} : F \subset \{\omega > M_{\text{out}}\} \text{ and } \omega(\mathbf{v}_{\text{in}}(F)) = M_{\text{out}}\} \neq \emptyset$$

and

$$\gamma(F) = \mathbf{v}_{\text{out}}(F) \quad \text{whenever } F \in \mathcal{F} \text{ and } \gamma \in \Gamma_{\text{special}}$$

Proof. Let $M = \max\{\omega(v) : v \in V\}$ and let $X = \cup\{\tau(E) : E \in \mathcal{O}\}$. Since $E \subset \{\omega \leq M\}$ for each $E \in \mathcal{O}$ we find that $X \subset \{\omega \leq M\}$.

Suppose $E \in \mathcal{O}$. Were it the case that $E \subset \{\omega = M\}$ we would have either $\sigma[E] \cap \{\omega > M\} \neq \emptyset$ or $\sigma^{-1}[E] \cap \{\omega > M\} \neq \emptyset$. Thus E meets $\{\omega = M\}$ in a vertex of E. It follows that

$$X \cap \{\omega = M\} \subset \mathcal{V}_{\text{in}} \cup \mathcal{V}_{\text{out}} \quad \text{and} \quad E \cap \{\omega < M\} \neq \emptyset \quad \text{for } E \in \mathcal{O}. \tag{6}$$

$\mathbb{R}^2 \sim Z_{1/2}$, respectively. Since Z_0 and Z_1 are connected and since any path starting on Z_0 and ending on Z_1 must pass through $Z_{1/2}$ we find that

$$\textit{either}\ \text{(iii)} \quad Z_0 \subset W_u \quad \text{and} \quad Z_1 \subset W_b \quad \textit{or}\ \text{(iv)} \quad Z_1 \subset W_u \quad \text{and} \quad Z_0 \subset W_b.$$

Since $V \cap \mathcal{V}_{\text{out}} \subset Z_0$ and $V \cap \mathcal{V}_{\text{in}} \subset Z_1$ we find that

$$\text{(v)} \qquad V \cap \mathcal{V}_{\text{out}} \subset W_u \quad \text{and} \quad V \cap \mathcal{V}_{\text{in}} \subset W_b \quad \text{in case (iii) holds.}$$

and that

(vi) $V \cap \mathcal{V}_{\rm in} \subset W_u$ and $V \cap \mathcal{V}_{\rm out} \subset W_b$ in case (iv) holds

It follows from (6) that $Z_{1/2} \subset \{\omega < M\}$; this implies $\{\omega \geq M\} \subset W_u$. Keeping in mind (v) and (vi) we find that

(vii) $V \cap \{\omega = M\} \subset \mathcal{V}_{\rm out}$ if (iii) holds.

and that

(viii) $V \cap \{\omega = M\} \subset \mathcal{V}_{\rm in}$ if (iv) holds

It follows that

$M_{\rm in} < M_{\rm out} = M$ in case (iii) holds and $M_{\rm out} < M_{\rm in} = M$ in case (iv) holds

Let

$$m = \begin{cases} M_{\rm in} & \text{if (iii) holds,} \\ M_{\rm out} & \text{if (iv) holds.} \end{cases}$$

Let

$$\mathcal{N} = \{F \in \mathcal{O} : E \cap \{\omega > m\} \neq \emptyset\}.$$

Lemma 5.1. Suppose $v \in V \cap \{\omega = m\}$, $E \in \mathcal{E}$, $v \in \mathbf{v}(E)$ and $E \subset \{\omega > m\}$. If *either* (iii) holds and $v \in \mathcal{V}_{\rm in}$ *or* (iv) holds and $v \in \mathcal{V}_{\rm out}$ then $E \in \mathcal{N}$.

Proof. Suppose $v \in V \cap \mathcal{V}_{\rm in} \cap \{\omega = m\}$, (iii) holds, $E \in \mathcal{E}$, $v \in \mathbf{v}(E)$ and $E \subset \{\omega > m\}$ but, contrary to the Lemma, $E \notin \mathcal{N}$. Then $\mathbf{v}(E) \subset \mathcal{V}_{\rm in}$ and, since $v \in Z_1 \subset W_b$ and W_b is open and connected, we have $E \subset W_b$. Let F be a sequence in $\mathcal{E}$ such that $F_0 = E$; $\mathbf{v}(F_{i-1}) \cap \mathbf{v}(F_i) \neq \emptyset$ whenever $i \in \mathbb{N}^+$; and $\mathbb{N} \ni \nu \mapsto \max \omega[F_\nu]$ increases to ∞ as $\nu \to \infty$. Since W_b is bounded there must be some $N \in \mathbb{N}^+$ such that $F_N \not\subset W_b$ and $F_i \subset W_b$ whenever $i \in \mathbb{N}$ and $0 \leq i < N$. Thus F_N meets Z_1 and Z_0 and this implies $F_N \in \mathcal{O}$ so $\mathbf{v}_{\rm in}(F_N) \in V \cap \mathcal{V}_{\rm in}$ but $\omega(\mathbf{v}_{\rm in}(F_N)) > \max \omega[F_{N-1}] > m$.

By a similar argument one deals with the other case. □

Suppose v and E are as in the preceding Lemma. Let $E_i = \sigma^i[E]$ for $i \in \mathbb{Z}$. Choose integers I, J such that $I \leq 0 \leq J$ and $E_i \in \mathcal{N}$ whenever $i \in \mathbb{Z}$ and $I \leq i \leq J$ but such that neither E_{I-1} nor E_{J+1} belong to $\mathcal{N}$. It follows that $\omega(\gamma(E_{I-1})) \leq m$ and $\omega(\gamma(E_{J+1})) \leq m$. Since $\omega(\gamma(E)) \geq m$ there are integers I', J' such that $I \leq I' \leq 0 \leq J' \leq J$ and distinct points a, b such that $\omega(a) = m = \omega(b)$, $a \in \mathbf{c}(\gamma(\sigma[E_{I'-1}]), \gamma(\sigma[E_{J'+1}]))$. It follows that $\gamma(E) = v$.

□

Corollary 5.1. Suppose E is degenerate and v is the vertex of E such that, according to Proposition 4.8, is such that $\cap\{\mathbf{v}(F) : F \in \mathbf{o}(E)\} = \{v\}$. Then

$$\gamma(F) = v \quad \text{whenever } \gamma \in \Gamma_{\rm special} \text{ and } F \in \mathbf{o}(E).$$

Proof. Suppose $\omega \in \mathbb{R}_2$. Then there will always be $w \in F \in \mathbf{o}(E)$ such that $\omega(w) > \omega(v)$ so our assertion follows directly from the preceding Theorem. □

6. The basic construction.

Let
$$\mathcal{P} = \{(a, E) : E \in \mathcal{E}_{\text{bdry}} \text{ and } \ a \in \mathbf{v}(E)\}.$$

Our main goal in this section is to provide an algorithm for computing a function P as in the following Theorem.

Theorem 6.1. There is a function
$$P : \mathcal{P} \to \mathcal{P}$$
such that if $(a, E) \in \mathcal{P}$ and $(b, F) = P(a, E)$ then there is $J \in \mathbb{N}^+$ such that
$$F = \sigma^J[E]$$
and such that, whenever $\gamma \in \Gamma_{\text{special}}$ and $\gamma(E) = a$, then
$$\gamma(F) = b \quad \text{and} \quad \{\gamma(\sigma^j[E]) : j \in \mathbb{I}(0, J) \subset \mathbf{c}(a, b). \tag{7}$$

Theorem 6.2. Suppose $\gamma_i \in \Gamma_{\text{special}}$ for $i \in \{1, 2\}$ Then $\gamma_1 \approx \gamma_2$.

Proof. Suppose $\mathcal{O} \in \mathbf{O}$. By Theorem 5.1 there is $E \in \mathcal{O}$ are such that $\gamma_1(E) = a = \gamma_2(E)$. Applying the previous Theorem repeatedly we obtain $b : \mathbb{N} \to \mathcal{V}$ and $\lambda : \mathbb{N} \to \mathbb{N}$ be such that $b_0 = a$, $\lambda_0 = 0$ and such that if $j \in \mathbb{N}$ then
$$(b_{j+1}, \sigma^{\lambda_{j+1}}[E]) = P(b_j, \sigma^{\lambda_j}[E]),$$
$$b_{j+1} = \gamma_i(b_j) \quad \text{for } i \in \{1, 2\},$$
and
$$\{\gamma_i(\sigma^k[E]) : k \in \mathbb{I}(\lambda_j, \lambda_{j+1}) \subset \mathbf{c}(b_j, b_{j+1}) \quad \text{for } i \in \{1, 2\}.$$
It follows that
$$\cup\{\mathbf{c}(\gamma_1(\sigma^k[E], \sigma^{k+1}[E])) : k \in \mathbb{N}\} = \cup\{\mathbf{c}(\gamma_2(\sigma^k[E], \sigma^{k+1}[E])) : k \in \mathbb{N}\}.$$
This in turn implies that $\mathbf{p}(\gamma_1) = \mathbf{p}(\gamma_2)$ so that, by Theorem 4.1, $\gamma_1 \approx \gamma_2$. □

6.1. The sets $\mathbf{w}_a(B)$.

Out construction will make use of these sets.

Definition 6.1. Whenever $a \in \mathbb{R}^2$ and $B \subset \mathbb{R}^2$ we let
$$\mathbf{w}_a(B) = \{a + t(x - a) : 0 < t < \infty \text{ and } x \in B\}.$$

We fix
$$a \in \mathbb{R}^2.$$

The following four Propositions are geometrically obvious; we leave their proofs to the reader.

Proposition 6.1. Suppose $\mathcal{E}$ is a finite subfamily of of $\mathbf{E}$, $V = \cap\{\mathbf{w}_a(E) : E \in \mathcal{E}\}$ and $\mathbf{int}\, V \neq \emptyset$.

There are $b, c \in \cup\{\mathbf{v}(E) : E \in \mathcal{E}\}$ such that $[a, b, c] > 0$ and $V = \mathbf{w}_a(\mathbf{c}(b, c))$.

Moreover, for each $E \in \mathcal{E}$ there are $d, e \in E$ such that $[a, d, e] > 0$ and
$$W = \mathbf{w}_a(\mathbf{c}(d, e)).$$

Proposition 6.2. Suppose $E \in \mathbf{E}$, $\mathbf{int}\, \mathbf{w}_a(E) \neq \emptyset$ and b, c are such that $\mathbf{v}(E) = \{b, c\}$ and $[a, b, c] > 0$. Then $\mathbf{w}(E) = \mathbf{h}_+(a, b) \cap \mathbf{h}_-(a, c)$.

Proposition 6.3. Suppose $b, c \in \mathbb{R}^2$, $[a, b, c] > 0$, $E = \mathbf{c}(b, c)$, $H \in \mathbf{H}$, $E \subset \mathbf{bdry}\, H$ and $a \notin H$. Suppose $F \in \mathbf{E}$ is such that $E \subset H$ and $F \cap \mathbf{int}\, \mathbf{w}_a(E) = \emptyset$. Then *either* $F \subset \mathbf{h}_-(a, b)$ *or* $F \subset \mathbf{h}_+(a, c)$.

Proposition 6.4. Suppose

(i) $E \in \mathbf{E}$, $\mathbf{int}\, \mathbf{w}_a(E) \neq \emptyset$, $H \in \mathbf{H}$, $E \subset \mathbf{bdry}\, H$ and $a \notin H$;
(ii) $F \in \mathbf{E}$, $\mathbf{v}(E) \cap \mathbf{v}(F) = \{e\}$ for some $e \in \mathbb{R}^2$ and $F \sim \{e\} \subset \mathbf{int}\, H$;
(iii) $I \in \mathbf{H}$, $F \subset \mathbf{bdry}\, I$ and $E \sim \{e\} \subset \mathbf{int}\, I$;
(iv) $\mathbf{int}\, V \neq \emptyset$ where we have set $V = \mathbf{w}_a(E) \cap \mathbf{w}_a(F)$.

Then

(v) $a \in \mathbf{int}\, I$;
(vi) for each $x \in V$ there exist unique $s, t \in (0, \infty)$ such that $a + s(x - a) \in E$ and $a + t(x - a) \in F$;
(vii) if x, s, t are as in (vi) then $s \leq t$ with equality only if $a + s(x - a) = e = a + t(x - a)$.

6.2. **The construction of P and the proof of Theorem 6.1.** Suppose $(a, E) \in \mathcal{P}$.

If E is degenerate we let $P(a, E) = (a, \sigma[E])$. If $\gamma \in \Gamma_{\text{special}}$ we infer from Proposition 4.8 that $\gamma(F) = a$ for all $F \in \mathbf{o}(E)$ so (7) holds.

So suppose E is not degenerate. Let $E_i = \sigma^i[E]$ for $i \in \mathbb{Z}$ and let

$$I = \max\{i \in \mathbb{N} : a \in \mathbf{v}(E_i)\}.$$

For each $i \in \mathbb{N}$ with $i > I$

$$W_i = \mathbf{w}_a(E_i) \quad \text{and we let} \quad V_i = \cap_{j=I+1}^{i} W_j.$$

Let

$$\mathcal{I} = \{i \in \mathbb{N} : i > I \text{ and } \mathbf{int}\, V_i \neq \emptyset\}.$$

Proposition 6.5. There is a positive integer $J \geq I + 1$ such that $\mathcal{I} = \mathbb{I}(I + 1, J)$. Moreover, $E_i \neq E_j$ whenever $i, j \in \mathbb{I}(I, J)$ and $i < j$.

Proof. Let $T \in \mathcal{T}_{\text{bdry}}$ be such that $\mathbf{e}(T) = \{E_I, E_{I+1}\}$ Since $T \subset W_{I+1} = V_{I+1}$ we find that $I + 1 \in \mathcal{I}$.

If N is the number of edges in $\mathbf{o}(E)$ and $m \in \mathbb{Z}$ then $\sigma^{mN}[E] = E$ and, therefore, $\mathbf{int}\, V_{mN} = \emptyset$ if $mN \geq I + 1$. This implies $\mathcal{I}$ is bounded.

Let $J = \max \mathcal{I}$. If $i, j \in \mathbb{N}^+$, $I + 1 \leq j < i$ and $i \in \mathcal{I}$ then $j \in \mathcal{I}$ since $V_i \subset V_j$. Thus $\mathcal{I} = \mathbb{I}(I + 1, J)$. □

By Proposition 6.1 there are for each $i \in \mathcal{I}$ unique points $r_i, s_i \in E_i$ such that $[a, r_i, s_i] > 0$ and

$$V_J = \mathbf{h}_+(a, r_i) \cap \mathbf{h}_-(a, s_i) = \mathbf{w}_a(\mathbf{c}(r_i, s_i)).$$

Let

$$R = \{a + t(r_J - a) : 0 < t < \infty\} \quad \text{and let} \quad S = \{a + t(s_J - a) : 0 < t < \infty\}.$$

(Of course $R = \{a + t(r_i - a) : 0 < t < \infty\}$ and $S = \{a + t(s_i - a) : 0 < t < \infty\}$ for any $i \in \mathcal{I}$.) By Prop 6.3 we have

$$\textit{either } (\text{I}) \quad E_{J+1} \subset \mathbf{h}_-(a, r_J) \quad \textit{or } (\text{II}) \quad E_{J+1} \subset \mathbf{h}_+(a, s_J).$$

Let

$$B = \{i \in \mathcal{I} : r_i \in \mathbf{v}(E_i)\}$$

and let

$$C = \{i \in \mathcal{I} : s_i \in \mathbf{v}(E_i)\}$$

From Proposition 6.1 we infer that neither B nor C is empty and that

$$V_J = \mathbf{w}_a(\mathbf{c}(r_i, s_j)) \quad \text{whenever } i \in B \text{ and } j \in C.$$

We let

$$K = \begin{cases} \max B & \text{in case (I) holds,} \\ \max C & \text{in case (II) holds.} \end{cases}$$

and we let

$$P(a, E) = \begin{cases} (r_K, E_K) & \text{in case (I) holds,} \\ (s_K, E_K) & \text{in case (II) holds.} \end{cases}$$

In case (I) holds we let $\omega \in \mathbb{R}_2$ be such that if $m = \omega(a)$ then $R \subset \{\omega = m\}$ and $S \subset \{\omega > m\}$. In case (II) holds we let $\omega \in \mathbb{R}_2$ be such that if $m = \omega(a)$ then $S \subset \{\omega = m\}$ and $R \subset \{\omega > m\}$.

We have

(8) $$E_i \cap \{\omega = m\} \neq \emptyset \quad \text{and} \quad E_i \cap \{\omega > m\} \neq \emptyset \quad \text{for } i \in \mathbb{I}(I, J).$$

Now suppose γ is special and $\gamma(E) = a$.

From Proposition 4.14 we infer that

(9) $$\gamma(E_i) = a \quad \text{for } i \in \mathbb{I}(0, I-1) \quad \text{which implies} \quad a \in \mathbf{c}(\gamma(E_{I-1}), \gamma(E_I)).$$

Suppose (I) holds. Since $r_K \in R \cap \mathbf{v}(E_K)$ and E_K meets the interior of V_K we find that $\gamma(E_K) \in \mathbf{h}_+(a, r_K) = \mathbf{h}_+(a, r_J)$. Since $\gamma(E_{J+1}) \in E_{J+1} \subset \mathbf{h}_-(a, r_K)$ we infer that for some $L \in \mathbb{I}(K, J)$ the segment $\mathbf{c}(\gamma(E_L), \gamma(E_{L+1}))$ meets R in a point c. Applying Theorem 4.2 with I, J there equal I, L and a, b there equal a, c we find that $\omega(\gamma(E_i)) = m$ for $i \in \mathbb{I}(I, K)$ and that $\gamma(E_K) = r_K$.

Suppose (II) holds. Since $s_K \in S \cap \mathbf{v}(E_K)$ and E_K meets the interior of V_K we find that $\gamma(E_J) \in \mathbf{h}_-(a, s_J) = \mathbf{h}_-(a, s_I)$. Since $\gamma(E_{J+1}) \in E_{J+1} \subset \mathbf{h}_+(a, s_K)$ we infer that some some $L \in \mathbb{I}(K, J)$ the segment $\mathbf{c}(\gamma(E_L), \gamma(E_{L+1}))$ meets R in a point c. Applying Theorem 4.2 with I, J there equal I, L and a, b there equal a, c we find that $\omega(\gamma(E_i)) = m$ for $i \in \mathbb{I}(I, K)$ and that $\gamma(E_K) = s_K$.

6.3. **Computational complexity.** *Now let us suppose that $\mathcal{V} = A[\mathbb{Z}^2]$ for some affine isomorphism A of $\mathbb{R}^2$.*

We will show that

(10) $$(J - I) \leq 3(K - I)$$

Let

$$U = \{a + t(x - a) : 0 < t \leq 1 \text{ and } x \in \mathbf{c}(r_J, s_J) \sim \{r_J, s_J\}\} = \mathbf{int}\, V_J.$$

Proposition 6.6. For any $E \in \mathcal{E}$ we have

$$E \cap S \neq \emptyset \text{ and } E \cap U \neq \emptyset \;\Leftrightarrow\; E = E_i \text{ for some } i \in \mathcal{I} \;\Leftrightarrow\; E \cap S \neq \emptyset \text{ and } E \cap U \neq \emptyset.$$

Proof. Suppose $F \in \mathcal{E}$, $F \cap R \neq \emptyset$ and $F \cap U \neq \emptyset$. Then there are $s \in (0, 1]$ and $x \in \mathbf{c}(r_J, s_J) \sim \{r_J, s_J\}$ such that $a + s(x - a) \in F$. For each $i \in \mathcal{I}$ we let t_i be such that $e_i = a + t_i(x - a) \in E_i$; by Lemma 4.15 (viii) we have $t_j < t_k$ whenever $j, k \in \mathcal{I}$ and $j < k$. We must have $s = t_i$ for some $i \in \mathcal{I}$ since otherwise F would meet the interior of the triangle T such that $\{E_i, E_{i+1}\} \in \mathbf{e}(T)$ for some $i \in \mathcal{I} \sim \{I\}$. Thus $a + t_i(x - a) \in E_i \cap F$ for some $i \in \mathcal{I}$. Since $e_i \notin \mathbf{v}(E_i)$ we find that $E = E_i$.

By a similar argument one shows that if $F \in \mathcal{E}$, $F \cap R \neq \emptyset$ and $F \cap U \neq \emptyset$ then $E = E_i$ for some $i \in \mathcal{I}$.

It follows directly from definitions that if $i \in \mathcal{I}$ then $E_i \cap R \cap S \cap U \neq \emptyset$. □

Suppose A is an orientation preserving affine isomorphism of $\mathbb{R}^2$ and P_A is the function arising from the construction just described with a, $\mathcal{T}$, $\mathcal{V}_{\rm in}$ replaced by $A(a)$, $\{A[T] : T \in \mathcal{T}\}$, $\{A(v) : v \in \mathcal{V}_{\rm in}\}$, respectively. Then

$$P_A(A(a), A[E]) = (A(b), A[F]) \quad \text{whenever } (a, E), (b, F) \in \mathcal{P}, \text{ and } (b, F) = P(a, E).$$

It follows that we may assume without loss of generality that

$$\mathcal{V} = \mathbb{Z}^2, \qquad a = 0.$$

It follows that for any $E \in \mathcal{E}$ there is one and only $\lambda(E) = (\lambda_1(E), \lambda_2(E)) \in \mathbb{Z}^2$ such that if $z = \lambda(E)$ then exactly one of the following holds:

(i) $E = \mathbf{c}(z, z + \mathbf{e}_1)$;
(ii) $E = \mathbf{c}(z, z + \mathbf{e}_2)$;
(iii) $E = \mathbf{c}(z + \mathbf{e}_2, z + \mathbf{e}_1)$;
(iv) $E = \mathbf{c}(z, z + \mathbf{e}_1 + \mathbf{e}_2)$.

Suppose (I) holds. Since $B \neq \emptyset$ there will exist $d = (d_1, d_2) \in \mathcal{V} \sim \{0\}$ such that

$$R \cap \mathcal{V} = \{wd : w \in \mathbb{N}^+\}.$$

Let $M \in (0, \infty)$ be such that $r_J = Md$. Let $L \in \mathbb{N}$ be such that $L \leq M < L + 1$.

Applying counterclockwise rotation by $n\pi/2$ radians for some $n \in \mathbb{N}^+$ if necessary we may assume without loss of generality that

$$d_1 > 0 \quad \text{and} \quad d_2 \geq 0.$$

Let

$$m = \frac{d_2}{d_1}$$

We suppose $0 \leq m \leq 1$ in proving (10) and leave it to the reader to carry out modify what we do below in a straightforward way to deal with case $1 < m$.

Proposition 6.7. $B = \{wd : w \in \mathbb{N}^+ \text{ and } w \leq L\}$. Also, $r_K = Ld$, $\lambda(E_K) = Ld$ and $\lambda_1(E_J) < (L + 1)d_1$.

Proof. Suppose $w \in \mathbb{N}^+$ and $w \leq L$. Let $v = wd$. If $w = M$ then $v = b_I \in B$ so suppose $w < M$.

Since $\mathbf{int}\, V_I \neq \emptyset$ there is $F \in \mathcal{E}$ such that $v \in \mathbf{v}(F)$ and $F \cap U \neq \emptyset$. By Lemma 6.6 we have $F = E_i$ for some $i \in \mathcal{I}$. Thus $v \in B$.

Let $F = \mathbf{c}(Ld, Ld + \mathbf{e}_1)$ and $G = \mathbf{c}(Ld, Ld + \mathbf{e}_1 + \mathbf{e}_2)$. Since $0 < d_2 \leq d_1$ we find that F and G meet U and that $\mathbf{c}(Ld, Ld + \mathbf{e}_1)$ does not meet U. From Proposition 4.15 (viii) and Proposition 6.6 we find that

$$E_K = \begin{cases} F & \text{if } G \notin \mathcal{E}, \\ G & \text{if } G \in \mathcal{E}. \end{cases}$$

it follows that $\lambda(E_K) = Ld$.

Since $r_J = Md$ and $M < L + 1$ we find that $\lambda_1(E_J) \leq Md_1 < (L + 1)d_1$. □

Lemma 6.1. Suppose $i, j \in \mathcal{I}$ and $i < j$. Then

$$\lambda_1(E_i) \leq \lambda_1(E_j).$$

Proof. Let $x = (x_1, x_2) \in E_j$ be such that $x_1 = \lambda_1(E_j)$ and let $t \in (0, \infty)$ be such that $tx \in E_i$. Then $t \leq 1$ by Lemma 4.15 (viii). Thus

$$\lambda(E_i) \leq tx_1 \leq x_1 = \lambda(E_j).$$

□

Let

$$\begin{aligned}\mathcal{K} &= \{k \in \mathbb{N} : k \le \lambda_1(E_I)\};\\ \mathcal{K}' &= \{k \in \mathcal{K} : \text{for some } w \in [k, k+1],\ mw \in \mathbb{N}\};\\ \mathcal{K}'' &= \mathcal{K} \sim \mathcal{K}'.\end{aligned}$$

For each $k \in \mathcal{K}$ let

$$N(k) = \{i \in \mathcal{I} : \lambda_1(E_i) = k\}.$$

Proposition 6.8. Suppose $k \in \mathcal{K}'$. Then $2 \le N(k) \le 4$.

Proof. Suppose $0 < m \le 1$. Let $l \in \mathbb{N}$ be such that $l \le mw < l+1$. Let $a = (k,l)$, $b = (k, l+1)$, $c = (k, l+2)$, $d = (k+1, l)$, $e = (k+1, l+1)$ and let

$$\mathcal{F} = \{\mathbf{c}(a,b), \mathbf{c}(b,e)\} \quad \text{and let} \quad \mathcal{G} = \{\mathbf{c}(b,d), \mathbf{c}(c,e)\}.$$

Then $\mathcal{F} \cup \mathcal{G}$ is the set of $F \in \mathcal{E}$ such that $\lambda_1(F) = k$ and $F \cap R \ne \emptyset$. Keeping in mind Proposition 6.6 we find that

$$\mathcal{F} \subset \{i \in \mathcal{I} : \lambda_1(E_i) = k\} \subset \mathcal{F} \cup \mathcal{G}.$$

We leave it to the reader to verify that $N(k) = 2$ in case $m = 0$. □

Proposition 6.9. Suppose $k \in \mathcal{K}''$. Then $N(k) = 3$.

Proof. Note that $0 < m < 1$ and let $l \in \mathbb{N}$ be such that $l \le mk < l+1$. Then $l < mk$ and $l < m(k+1) < l+1$. Let $a = (k,l)$, $b = (k, l+1)$, $c = (k+1, l)$, $d = (k+1, l+1)$ and let

$$\mathcal{F} = \{\mathbf{c}(a,b), \mathbf{c}(a,d), \mathbf{c}(b,c), \mathbf{c}(b,d).$$

Then $\mathcal{F}$ is the set of $F \in \mathbf{E}$ such that $\lambda_1(F) = k$ and $F \cap R \ne \emptyset$. Keeping in mind Proposition 6.6 and the fact that exactly one of $\mathbf{c}(a,d)$ and $\mathbf{c}(b,c)$ belongs to $\mathcal{E}_{\text{bdry}}$ we find that $N(k) = 3$. □

$$\begin{aligned}K - I &\ge 1 + \sum_{k \in \mathcal{K}',\ k \le Ld_1} N(k) + \sum_{k \in \mathcal{K}'', k \le Ld_1} N(k)\\ &\ge 1 + 2Ld_2 + 3L(d_1 - d_2)\\ &= 1 + L(3d_1 - d_2).\end{aligned}$$

$$\begin{aligned}(J - I) - (K - I) &\le \sum_{k \in \mathcal{K}',\ Ld_1 < k < (L+1)d_1} N(k) + \sum_{k \in \mathcal{K}''\ \ Ld_1 < k < (L+1)d_1} N(k)\\ &\le 4d_2 + 3(d_1 - d_2)\end{aligned}$$

Thus

$$\begin{aligned}\frac{J - I}{K - I} &\le 1 + \frac{J - I) - (K - I)}{K - I}\\ &\le 1 + \frac{(3d_1 + d_2)}{1 + L(3d_1 - d_2)}\\ &\le 1 + \frac{4d_1}{1 + L(2d_1)}\\ &\le 1 + \frac{4d_1}{1 + 2d_1} \le 3.\end{aligned}$$

7. Some theorems on plane curves.

Throughout this section we fix

$$R \in (0, \infty).$$

Definition 7.1. For each $c \in \mathbb{R}^2$ and $v \in \mathbb{S}^1$ we define

$$\mathbf{D}(c, v, R)$$

as follows. We let $\mathbf{D}(0, \mathbf{e}_2, R)$ equal

$$\left\{ x \in \mathbb{R}^2 : |x_1| < R \text{ and } -R + \sqrt{R^2 - x_1^2} \le x_2 \le R - \sqrt{R^2 - x_1^2} \right\}$$

and if $(c, v) \neq (0, \mathbf{e}_2)$ we let $\mathbf{D}(c, v, R) = \rho[\mathbf{D}(0, \mathbf{e}_2, R)]$ where ρ is the rigid motion of $\mathbb{R}^2$ which carries 0 to c, $\mathbf{e}_1$ to $c+v^\perp$ and $\mathbf{e}_2$ to $c+v$. Alternatively, $\mathbf{D}(c, v, R)$ is the bounded connected component of the complement in $\{x \in \mathbb{R}^2 : |(x - c) \bullet v^\perp| < R\}$ of $\mathbf{U}(c + Rv, R) \cup \mathbf{U}(c - Rv, R)$.

Definition 7.2. Suppose $a, b \in \mathbb{R}^2$, $0 < R < \infty$ and $0 < |a - b| < 2R$. We let

$$\mathbf{c}_+(a, b, R) \quad \text{and} \quad \mathbf{c}_-(a, b, R)$$

be the points on the perpendicular bisector of $\mathbf{c}(a, b)$ such that

$$|a - \mathbf{c}_\pm(a, b, R)| = R = |b - \mathbf{c}_\pm(a, b, R)|$$

and whose inner products with $(b - a)^\perp$ are positive and negative, respectively.

We let

$$\mathbf{L}(a, b, R) = \mathbf{B}(\mathbf{c}_+(a, b, R), R) \cap \mathbf{B}(\mathbf{c}_-(a, b, R), R).$$

For $e \in \{a, b\}$ we let

$$\mathbf{W}_e(a, b, R) = \{t(x - e) : x \in \mathbf{L}(a, b, R)\}.$$

Proposition 7.1. Suppose $a, b \in \mathbb{R}^2$, $0 < |a - b| < 2R$, $u \in \mathbb{S}^1 \cap \mathbf{W}_a(a, b, R)$ and $v \in \mathbb{S}^1 \cap \mathbf{W}_b(a, b, R)$. Then

$$|u + v| \le \frac{|a - b|}{R}.$$

Proof. Let $A = \mathbb{S}^1 \cap \mathbf{W}_a(a, b, R)$ and let $B = \mathbb{S}^1 \cap \{t(x - \mathbf{c}_-(a, b, R)) : x \in \mathbf{c}(a, b)\}$. Now $-v \in \mathbf{W}_a(a, b, R)$ so $|u + v|$ does not exceed the diameter of A. Moreover A is congruent to B the diameter of which equals $|a - b|/R$. □

Lemma 7.1. Suppose $|a - b| < 2R$. Then $\mathbf{L}(a, b, R) \subset \mathbf{B}((1/2)(a + b), |a - b|/2)$. In particular, $\mathbf{diam}\, \mathbf{L}(a, b, R) = |a - b|$.

Proof. Exercise for the reader. □

Definition 7.3. We let

$$\mathcal{P}(R)$$

be the set of ordered pairs (I, P) such that

(i) I is a nonempty open interval;
(ii) $P : I \to \mathbb{R}^2$;
(iii) P is continuously differentiable and $|P'(s)| = 1$ for $s \in I$;
(iv) $\limsup_{t \to s} |P'(t) - P'(s)|/|t - s| \le 1/R$ whenever $s \in I$;

Remark 7.1. Suppose $(I,P) \in \mathcal{P}(R)$ and $s_* \in \{\inf I, \sup I\} \sim \{-\infty, \infty\}$. Owing to (iii) and (iv) in the preceding definition we find that the limits

$$\lim_{I \ni s \to s_*} P(s) \quad \text{and} \quad \lim_{I \ni s \to s_*} P'(s)$$

exist.

Lemma 7.2. Suppose $0 < R < \infty$, $c \in \mathbb{R}^2$, I is an open interval and

$$P(s) = c + R\mathbf{u}(s/R) \quad \text{for } s \in I \text{ and} \quad C = \{P(s) : s \in I\}$$

Then $(I,P) \in \mathcal{P}(R)$. Moreover, the diameter of the range of P is less than $2R$ if and only if the length of I is less than πR.

Proof. Obvious. □

Theorem 7.1. Suppose

(i) $(I,P) \in \mathcal{P}(R)$;
(ii) $s_* \in I$ and $c = P(s_*)$;
(iii) $u = P'(s_*)$, $v \in \mathbb{S}^1$, $u \bullet v = 0$ and

$$U(s) = (P(s) - c) \bullet u \quad \text{and} \quad V(s) = (P(s) - c) \bullet v \quad \text{for } s \in I;$$

(iv) I_* is the connected component of s_* in $\{s \in I : |U(s)| < R\}$ and $J_* = \{U(s) : s \in I_*\}$;
(v) $f = \{(U(s), V(s)) : s \in I_*\}$;

Then

(vi) I_* is an open interval, $s_* \in I_*$, J_* is an open interval and $0 \in J_* \subset (-R, R)$;
(vii) $f : J_* \to \mathbb{R}$, f is continuously differentiable and

$$\{P(s) : s \in I_* := \{c + tu + f(t)v : t \in J_*\};$$

(viii) $|f(t)| \le R - \sqrt{R^2 - t^2}$ whenever $t \in J_*$ and

$$\{P(s) : s \in I_*\} \subset \mathbf{D}(c, v, R);$$

(ix) $|f'(t)| \le |t|/\sqrt{R^2 - t^2}$ whenever $t \in J_*$;
(x) if $\{P(s) : s \in I \text{ and } s < s_*\} \sim \mathbf{D}(c, v, R) \neq \emptyset$ then $\inf J_* = -R$ and if $\{P(s) : s \in I \text{ and } s > s_*\} \sim \mathbf{D}(c, v, R) \neq \emptyset$ then $\sup J_* = R$;

Proof. (vi) is obvious.

Without loss of generality we may assume $R = 1$, $s_* = 0$, $c = 0$, $u = \mathbf{e}_1$ and $v = \mathbf{e}_2$. Let $Q = P'$.

Let I_{**} be the connected component of 0 in $\{s \in I_* : Q(s) \bullet \mathbf{e}_1 > 0\}$ and let $J_{**} = \{U(s) : s \in I_{**}\}$; and let $g = \{(U(s), V(s)) : s \in J_{**}\}$. Evidently, $I_{**} \subset I_*$ and $J_{**} \subset J_*$. Since $U'(s) = P'(s) \bullet u > 0$ for $s \in I_{**}$ we find that (vi) and (vii) hold with I_*, J_*, f replaced by I_{**}, J_{**}, g, respectively. Let $s_- = \inf I_* < 0 < \sup I_* = s_+$ and let $t_- = \inf J_* < 0 < \sup J_* = t_+$.

It follows that

$$Q(t, g(t)) = \mathbf{w}(g'(t))^{-1}(1, g'(t)) \quad \text{whenever } t \in J_*$$

where we have set $\mathbf{w}(m) = \sqrt{1 + m^2}$ for $m \in \mathbb{R}$.

Let $Q = P'$ and for each $s \in I$ let

$$\kappa(s) = \limsup_{h \to 0} \frac{|Q(s+h) \bullet Q(s)^\perp|}{|h|};$$

since $Q(t) \bullet Q(s)^\perp = (Q(t) - Q(s)) \bullet Q(s)^\perp$ whenever $s, t \in I$ we infer from (ii) that $|\kappa(s)| \le 1$ for $s \in I$.

Suppose $s \in J_{**}$, $0 < h < \infty$ and $s + h \in J_{**}$. We have

$$\frac{Q(s+h) \bullet Q(s)^\perp}{h} = A(h)B(h)C(h)$$

where we have set

$$\begin{aligned} A(h) &= \frac{g'(U(s+h)) - g'(U(s))}{U(s+h) - U(s)}; \\ B(h) &= \frac{U(s+h) - U(s)}{h}; \\ C(h) &= \frac{1}{\mathbf{w}(g'(U(s+h)))\mathbf{w}(g'(U(s)))}. \end{aligned}$$

Now

$$B(h)C(h) \to \frac{1}{\mathbf{w}(g'(U(s)))^3}$$

as $h \to 0$. Since $\mathbf{w}''(m) = 1/\mathbf{w}(m)^3$ for $m \in \mathbb{R}$ we find that

$$\mathbf{Lip}(\mathbf{w}' \circ g') \le 1.$$

Since $\mathbf{w}(0) = 0$ and $g'(0) = 0$ this implies that

$$|\mathbf{w}'(g'(x))| = |\mathbf{w}'(g'(x)) - \mathbf{w}'(g'(0))| \le |x| \quad \text{for } x \in J_{**};$$

since $\mathbf{w}'$ is increasing we find that

$$|g'(x)| \le |\mathbf{v}(x)| = \frac{|x|}{\sqrt{1-x^2}} \quad \text{whenever } x \in J \cap (-1,1) \tag{11}$$

where $\mathbf{v}$ is the function inverse to $\mathbf{w}'$. This in turn implies that

$$|g(x)| \le 1 - \sqrt{1-x^2} \quad \text{for } x \in J_{**} \cap (-1,1). \tag{12}$$

Thus (viii) and (ix) hold with J_*, f replaced by J_{**}, g, respectively, and the Theorem will be proved if we can show $J_{**} = J_*$. Suppose $x_0 = \sup I_{**} \in I_*$. From (viii) we infer that $|\lim_{t \uparrow t_0} g'(t)| < \infty$ which in turn implies that $y_0 = \lim_{t \uparrow t_0} g(t)$ exists and is finite. Thus $Q(x_0, y_0) \bullet \mathbf{e}_1 > 0$ which implies there is a larger open interval than I_{**} on which $Q \bullet \mathbf{e}_1 > 0$. Thus $\sup J_{**} = \sup J_*$ and, therefore, $\sup I_{**} = \sup I_*$. In a similar fashion one shows that $\inf J_{**} = \inf J_*$ and $\inf I_{**} = \inf I_*$. □

Theorem 7.2. Suppose $(I, P) \in \mathcal{P}(R)$; $\mathbf{diam}\, I < \infty$;

$$a = \lim_{s \downarrow \inf I} P(s) \quad \text{and} \quad b = \lim_{s \uparrow \sup I} P(s);$$

$$r = \frac{|a-b|}{2} < R \quad \text{and} \quad m = \frac{1}{2}(a+b);$$

$w \in \mathbb{S}^1$ and

$$a \bullet w < P(s) \bullet w < b \bullet w \quad \text{whenever } s \in I. \tag{13}$$

Then

$$\{P(s) : s \in I\} \subset \mathbf{B}(m, r).$$

Proof. We may suppose without loss of generality that $m = 0$ and $w = \mathbf{e}_1$. Let $\rho = \sup\{|P(s)| : s \in I\}$ and suppose, contrary to the Lemma, that $\rho > r$. Since $|a| < \rho$ and $|b| < \rho$ there is $s_* \in I$ such that $|P(s_*)| = \rho$ and $P(s_*) \bullet Q(s_*) = 0$. Let $v \in \mathbb{S}^1$ be such that $P(s_*) = \rho v$. Let $u = v^\perp$; then $Q(s_*) = \pm u$. Let $c = P(s_*)$ and let I_*, f, J_*, etc., be as in Theorem 7.1. Since $\mathbf{B}(0,r) \cap \mathbf{D}(c,v,R) = \emptyset$ we find that $(-R,R) \subset J_*$.

Suppose

$$u \bullet w > 0 \quad \text{and} \quad v \bullet w \geq 0. \tag{14}$$

Let

$$\zeta_\pm(t) = c \mp Rv + R(tu \pm \sqrt{1-t^2}v) \quad \text{for } 0 \leq t \leq 1,$$

let $A_\pm = \{\zeta_\pm(t) : 0 \leq t < 1\}$ and note that

$$A_+ \cup A_- = \{x \in \mathbf{bdry}\,\mathbf{D}(c,v,R) : c \bullet u \leq x \bullet u < R + c \bullet u\}.$$

We have

$$\zeta_+(0) \bullet w = ((c - Rv) + Rv) \bullet w = c \bullet w < b \bullet w$$

Since $w = (w \bullet u)u + (w \bullet v)v$ we find in view of (14) that

$$(w \bullet u)u + \sqrt{1-(w \bullet u)^2}v = (w \bullet u)u + (w \bullet v)v = w$$

so that

$$\zeta_+(w \bullet u) = ((c - Rv) + Rw) \bullet w = \rho v \bullet w + R(1 - v \bullet w) > r > b \bullet w$$

so that there is $t_* \in [0, w \bullet u)$ such that $\zeta_+(t_*) \bullet w = b \bullet w$. Thus for any $\lambda \in [0,1]$ we have

$$((1-\lambda)\zeta_+(t_*) + \lambda\zeta_-(t_*)) \bullet w = \zeta_+(t_*) \bullet w + 2\lambda R(1 - \sqrt{1-t_*^2})v \bullet w \geq b \bullet w.$$

Since $(-R,R) \subset J_*$ we find that $P(s) \in \mathbf{c}(\zeta_+(t_*), \zeta_-(t_*))$, contrary to (13). □

Theorem 7.3. Suppose $(I,P) \in \mathcal{P}(R)$ and

$$\mathbf{diam}\,\{P(s) : s \in I\} < 2R. \tag{15}$$

Then $\mathbf{diam}\, I \leq \pi R$ and

$$\{P(s) : s \in I\} \subset \mathbf{L}(a,b,R). \tag{16}$$

Moreover, if

$$a = \lim_{I \ni s \to \inf I} P(s) \quad \text{and} \quad b = \lim_{I \ni s \to \sup I} P(s)$$

and

$$t_a = \lim_{I \ni s \to \inf I} P'(s) \quad \text{and} \quad t_b = \lim_{I \ni s \to \sup I} P'(s)$$

then

$$|t_a - t_b| \leq \frac{|a-b|}{R}.$$

Proof. Suppose $s_0, s_1 \in I$ and $s_0 < s_1$. Let $s_* = (s_0 + s_1)/2$. Let $u = P'(s_*)$ and let $v = u^\perp$. If $i \in \{0,1\}$ and $P(s_i) \in \mathbf{D}(P(s_*), v, R)$ we infer from Theorem 7.1 (ix) that $|s_i - s_*| \leq \pi R/2$ so that $s_1 - s_0 \leq \pi R$.

Were it the case that $\{P(s_0), P(s_1)\} \cap \mathbf{D}(P(s_*), v, R) = \emptyset$ we could infer from Theorem 7.1 (x) that there would be $\tilde{s}_0 \in [s_0, s_*)$ and $\tilde{s}_1 \in (s_*, s_1]$ such that $(P(\tilde{s}_0) - P(s_*)) \bullet u = -R$ and $(P(\tilde{s}_1 - P(s_*)) \bullet u = R$ and this would imply $|P(\tilde{s}_1) - P(\tilde{s}_0| \geq 2R$, contrary (15).

It follows that $\mathbf{diam}\, I \le \pi R$. Keeping in mind Remark 7.1 we infer infer the existence of a, b, t_1 and t_b as in the statement of the Theorem.

We now prove (16). Let $C = \{P(s) : s \in I\}$ and let $d \in \{\mathbf{c}_+(a,b,R), \mathbf{c}_-(a,b,R)\}$. Suppose, contrary to (16), $C \not\subset \mathbf{B}(c,R)$. Let $c \in \mathbf{cl}\, C$ be such that $|x-d| \le |c-d|$ whenever $x \in C$. Let $S \in (0,\infty)$ and $w \in \mathbb{S}^1$ be such that $c - d = Sw$. Suppose, contrary to (16), $S > 1$. Since $c \notin \{a,b\}$ we have $c = P(s_*)$ for some $s_* \in I$ and $P'(s_*) = \pm w^\perp$. But $\mathbf{D}(d,v,R) \cap \mathbf{U}(c,R) = \emptyset$ so $\{a,b\} \cap \mathbf{D}(d,w,R) = \emptyset$. We infer from Theorem 7.1 (x) that there are $s_\pm \in I$ such that $|(P(s_\pm) \pm d) \bullet w^\perp| = R$ which implies $|P(s_+) - P(s_-)| \ge 2R$ which is contrary (15). Thus $C \subset \mathbf{B}(c,R)$ and, therefore, (16) holds. The final assertion of the Theorem now follows from Proposition 7.1. □

7.1. **Length.**

Theorem 7.4. Suppose $(I,P) \in \mathcal{P}(R)$ and

$$d = \frac{1}{2}\mathbf{diam}\,\{P(s) : s \in I\} < 2R.$$

Then

$$\mathbf{diam}\, I \le 2 \arcsin \frac{d}{R}$$

with equality if and only $\{P(s) : s \in I\}$ is a subset of a circle or radius R.

Proof. The hypotheses of Theorem 7.3 hold so $\mathbf{diam}\, I \le \pi R$ and we may let a, b, t_a, t_b be as in the statement of that Theorem. Let

$$f(s) = |(P(s) - a) \bullet P'(s)| - |(P(s) - b) \bullet P'(s)| \quad \text{for } s \in I.$$

It follows that

$$\lim_{s \downarrow \inf I} f(s) = -|(a-b) \bullet t_a| < 0 \quad \text{and} \quad \lim_{s \downarrow \sup I} f(s) = |(a-b) \bullet t_b| > 0$$

so there is $s_* \in I$ such that if $c = P(s_*)$ and $u = P'(s_*)$ then for some $r \in (0,\infty)$ we have

$$|(P(s_*) - x) \bullet u| = r \quad \text{for } x \in \{a,b\}.$$

Let $v = u^\perp$, let f, I_*, J_*, etc., be as in Theorem 7.1 and let $C = \{P(s) : s \in J_*\}$. Suppose $\sup J_* \le r$. Then $b \notin C$ so $\sup J_* = R \le r$. But this forces $a \notin C$ so $\inf J_* = -R$ and $\mathbf{diam}\, C \ge 2R$ which we have excluded so $(-r,r) \subset J_*$. From Theorem 7.1 (ix) we infer that the length of C does not exceed $2\arcsin(r/R) lambda(r)$ thus establishing our length estimate. We also find that equality holds in the length estimate if and only if *either* $f(t) = R - \sqrt{R^2 - t^2}$ for $t \in (-r,r)$ *or* $f(t) = -R + \sqrt{R^2 - t^2}$ for $t \in (-r,r)$ which is to say C is a subset of a circle of radius R. □

8. The open set Ω.

We assume that throughout this section that Ω is a bounded open subset of $\mathbb{R}^2$ whose boundary $\partial\Omega$ is a continuously differentiable embedded submanifold of $\mathbb{R}^2$ with length L.

We do *not* assume Ω is connected.

We let $T, N : \partial\Omega \to \mathbb{S}^1$ be such that N is the unit normal to $\partial\Omega$ which points out of Ω and $T = N^\perp$. We let

$$\rho(x) = \mathbf{dist}\,(x, \partial\Omega) \quad \text{for } x \in \mathbb{R}^2$$

and we let

$$h(y,r) = y + rN(y) \quad \text{for } (y,r) \in (\partial\Omega) \times \mathbb{R}.$$

Proposition 8.1. Suppose $0 < R < \infty$ and

$$U = \{x \in \mathbb{R}^2 : \rho(x) < R\}.$$

Then the following statements are equivalent.

(i) $\xi|U$ is a function.
(ii) $h|((\partial\Omega) \times (-R,R))$ is univalent.
(iii) $(\partial\Omega) \cap (\mathbf{B}(b + RN(b), R) \cup \mathbf{B}(b - RN(b), R)) = \{b\}$ whenever $b \in \partial\Omega$.
(iv) $|(y-b) \bullet N(b)| \le |y-b|^2/2R$ whenever $y, b \in \partial\Omega$.

Proof. We leave the proof as a straightforward exercise for the reader. We suggest showing (i) implies (iii) implies (ii) implies (i). That (iii) and (iv) are equivalent follows by directly calculation. □

We now assume that there are a positive real number R such that with $U = \{x \in \mathbb{R}^2 : \rho(x) < R\}$ the equivalent conditions of Proposition 8.1 hold.

Inequality (i) implies that the normal N is Lipschitzian. Our assumption about the reach of $\partial\Omega$ is global; in particular, if $\partial\Omega$ is twice differentiable and the absolute value of the curvature of $\partial\Omega$ at any point is less than $1/R$ for some positive real number R then our assumptions need not hold; consider

$$\Omega = \{x \in \mathbb{R}^2 : |x| \in \{R, R+h\}\}$$

where h is a small positive number; it is not too difficult to construct examples of this sort where $\partial\Omega$ is connected.

We assume that

$$\mathbf{diam}\, T < h < R \quad \text{whenever } T \in \mathcal{T}_{\text{bdry}}$$

where $\mathcal{T}$ is as in 4 and where

$$\mathcal{V}_{\text{in}} = \{v \in \mathcal{V} : v \in \Omega\}.$$

We will prove the following Theorem.

Theorem 8.1. Suppose $\gamma \in \Gamma_{\min}$. Then

$$\frac{R-h}{R} L \le \mathbf{l}(\gamma) \le L$$

where L is the length of $\partial\Omega$.

8.1. **More on the geometry of $\partial\Omega$.**

Lemma 8.1. The length of each connected component of $\partial\Omega$ is at least $2\pi R$.

Proof. Let C be a connected component of $\partial\Omega$ and let G be the bounded open subset of $\mathbb{R}^2$ with boundary C. Note that $G \subset \Omega$ or $G \cap \Omega = \emptyset$. Let

$$\zeta = \begin{cases} 1 & \text{if } G \subset \Omega, \\ -1 & \text{if } G \cap \Omega = \emptyset. \end{cases}$$

Suppose $u \in \mathbb{S}^1$. Since C is compact there is $a \in C$ be such that

$$\{x \bullet u : x \in C\} \le a \bullet u;$$

clearly, $N(a) = \zeta u$.

Thus $\{N(x) : x \in C\} = \mathbb{S}^1$. It follows that

$$2\pi \le \int_C |N'| \le \frac{L}{R}$$

where L is the length of C. □

Theorem 8.2. Suppose $a, b \in \partial\Omega$ and $0 < |a-b| < 2R$. Then there is one and only one connected component C of $(\partial\Omega) \sim \{a, b\}$ such that $\{a, b\} \in \mathbf{cl}\, C$ and whose length is less than πR. Moreover,

$$\text{(17)} \qquad \mathbf{diam}\, C = |a-b|, \quad C \subset \mathbf{L}(a, b, R) \quad \text{and} \quad |T(a) - T(b)| \le \frac{|a-b|}{R}.$$

Finally, if $X = \mathbf{c}(a, b)$ then $X \subset U$, $\xi|X$ is univalent and $\xi[X] = \mathbf{cl}\, C$.

Proof. Let m be the midpoint of line segment joining a to b. Then

$$\mathbf{dist}\,(m, \partial\Omega) \le |a-m| = |b-m| = \frac{|a-b|}{2} < R$$

so $m \in U$. Let $c = \xi(m)$, let $u = T(c)$ and let $v = N(c)$. We may assume without loss of generality that $c = 0$.

We also have

$$\text{(18)} \quad |e| \le |e-m| + |m| = \frac{|a-b|}{2} + \mathbf{dist}\,(m, \partial\Omega) = \frac{3R}{2} < 2R \quad \text{for } e \in \{a, b\}.$$

Let $S = \{x \in \mathbb{R}^2 : |x_1| < R\}$. If $e \in \{a, b\}$ we have

$$|e \bullet u| = |(e-m) \bullet u| \le |e-m| = \frac{|a-b|}{2} < R$$

so

$$\text{(19)} \qquad \{a, b\} \subset S.$$

Let D be the connected component of 0 in $\partial\Omega$ and let $h \in (0, \infty)$ be such that the length of D equals $2h$. Let $I, P, \tilde{c}$ be such that $I = (-h, h)$; $(I, P) \in \mathcal{P}(R)$; $P(0) = 0$; P is univalent; and $\{P(s) : s \in I\} = D \sim \{\tilde{c}\}$. Then $h \ge \pi R$ by Lemma 8.1. Let I_*, J_*, f, etc., be as in Theorem 7.1 with s_* there equal 0 and let $E = \{P(s) : s \in I_*\}$. By (viii) and (ix) of Theorem 7.1 $E \subset \mathbf{D}(0, v, R)$ and the length of E does not exceed πR. If $s \in I_*$ then $|s| < \pi R/2 \le h$ by Theorem 7.1 (viii). Thus neither $\{P(s) : -h < s < 0\}$ nor $\{P(s) : 0 < s < h\}$ is a subset of $\mathbf{D}(c, v, R)$. It follows from Theorem 7.1 that $I_* = (-R, R)$.

It follows from Theorem 7.2 that $C \subset \mathbf{B}(m, r)$. Thus $\mathbf{diam}\, C \le 2r < 2R$ so, by Theorem 7.3, $C \subset \mathbf{L}(a, b, R)$ and the assertions of (17) follow from Theorem ??.

Let $\eta(t) = (1-t)a + tb$ for $t \in [0, 1]$. Then

$$\mathbf{dist}\,(\eta(t), \partial\Omega) \le \min\{|\eta(t) - a|, |\eta(t) - b|\} \le \frac{|a-b|}{2} < R$$

for any $t \in [0, 1]$ so $X \subset U$.

Suppose there were $s, t \in [0, 1]$ such that $s \ne t$ and $\xi(\eta(s)) = d = \xi(\eta(t))$. Then there would be $\rho, \sigma \in (-R, R)$ such that $\eta(s) = d + \rho N(d)$ and $\eta(t) = d + \sigma N(d)$. It would then follow that $\{a, b\} \subset \{d + zN(d) : z \in \mathbb{R}\}$. But since $\partial\Omega \cap (\mathbf{B}(d + RN(d), R) \cup \mathbf{B}(d - RN(d), R)) = \{d\}$ we would have that $|a-b| \ge 2R$. Thus $\xi \circ \eta$ is univalent so $\xi|X$ is univalent. Thus $\{\xi \circ \eta(t) : 0 < t < 1\}$ is a connected component of $(\partial\Omega) \sim \{a, b\}$ which contains $c = \xi(m) = \xi \circ \eta(1/2)$. It follows that $\xi[X] = \mathbf{cl}\, C$. □

Definition 8.1. Suppose $a, b \in \partial\Omega$ and $0 < |a - b| < 2R$. Keeping in mind the previous Theorem, we let

$$\mathbf{a}(a, b)$$

be the unique connected component of $\partial\Omega \sim \{a, b\}$ whose length is less than πR.

Remark 8.1. The following Theorem and its proof come from [**?**, 4.4(8)].

Theorem 8.3. Suppose $x, a \in U$ and $r = \max\{\rho(x), \rho(a)\}$. Then

$$|\xi(x) - \xi(a)| \leq \frac{R}{R - r}|x - a|.$$

Proof. Let $y = \xi(x)$ and let $b = \xi(y)$. From Proposition 8.1 (iv) we obtain

$$(x - y) \bullet (y - b) \geq -\frac{|y - b|^2 |x - y|}{2R} \geq -\frac{r|y - b|^2}{2R}$$

and

$$(a - b) \bullet (y - b) \geq -\frac{|y - b|^2 |a - b|}{2R} \geq -\frac{r|y - b|^2}{2R}.$$

Thus

$$\begin{aligned} |x - a||y - b| &\geq (x - a) \bullet (y - b) \\ &= [(y - b) + (x - y) + (b - a)] \bullet (y - b) \\ &\geq \left(1 - \frac{r}{2R} - \frac{r}{2R}\right)|y - b|^2 \\ &= \frac{R - r}{R}|y - b|^2. \end{aligned}$$

□

8.2. **Inscribing the polygon.**

Definition 8.2. Let $\beta \in \Gamma$ be such that for each $E \in \mathcal{E}_{\text{bdry}}$ we have $\beta(E) \in \partial\Omega$ and $\{(1 - t)\mathbf{v}_{\text{out}}(()E) + t\gamma(E) : 0 \leq t < 1\} \cap \partial\Omega = \emptyset$.

Theorem 8.4. Suppose $E \in \mathcal{E}_{\text{bdry}}$, $F = \sigma(E)$, $a = \gamma(E)$, $b = \gamma(F)$, $a \neq b$ and $u \in \mathbb{S}^1$ is such that $b - a = |b - a|u$. Then

$$|u - T(x)| < \quad \text{whenever } x \in \mathbf{a}(a, b).$$

Proof. Choose f, etc., as in ?? such that... We may assume without loss of generality that $a = 0$, $T(a) = -\mathbf{e}_1$ and $N(a) = \mathbf{e}_2$. Let $x \in (-R, R)$ be such that $b = (w, f(w))$. We will show that $w < 0$.

Case One. $\mathbf{v}_{\text{out}}(E) = \mathbf{v}_{\text{out}}(F)$**.** Suppose $\mathbf{v}_{\text{out}}(E) = (x, y)$. Then $ty > f(tx)$ whenever $0 < t \leq 1$. □

Lemma 8.2. (WRONG!) Suppose $E \in \mathcal{E}_{\text{bdry}}$, $a \in E \cap \partial\Omega$, $b \in \sigma[E] \cap \partial\Omega$ and $a \neq b$. Then $|a - b| < R$ and no vertex of $T = \mathbf{c}(E \cup \sigma[E])$ lies in $\mathbf{a}(a, b)$.

Proof. Suppose, contrary to the Lemma, c is a vertex of E which lay on the interior of $\mathbf{a}(a, b)$ relative to $\partial\Omega$. Keeping in mind ?? and using the Mean Value Theorem we obtain $a_*, b_* \in \mathbf{a}(a,b)$ such that $c - a = |a-c|T(a_*)$ and $b - c = |b-c|T(b_*)$ where we have interchanged a and b if necessary. But then, by ? of Theorem 7.1,

$$\begin{aligned}0 \le \frac{(c-a)\bullet(b-c)}{|c-a||b-c|} \\ &= T(a_*)\bullet T(b_*) \\ &= (T(a_*) - T(b_*))\bullet T(b_*) - 1 \\ &\le \frac{|a_* - b_*|}{R} - 1 \\ &\le \frac{|a-b|}{R} - 1\end{aligned}$$

which contradicts our hypothesis that $\mathbf{diam}\, T < R$. □

Lemma 8.3. Suppose γ is a choice function for $\mathcal{E}_{\text{bdry}}$ such that

$$\gamma(E) \in \partial\Omega \quad \text{for } E \in \mathcal{E}_{\text{bdry}}.$$

Then the family of the interiors of $\mathbf{a}(\gamma(E), \gamma(\sigma[E]))$ relative to $\partial\Omega$ corresponding to $E \in \mathcal{E}_{\text{bdry}}$ with $\gamma(E) \ne \gamma(\sigma[E])$ is disjointed.

In particular, $\mathbf{l}(\gamma)$ does not exceed the length of $\partial\Omega$.

Proof. Suppose the Lemma were false. Keeping in mind that $\mathbf{a}(\gamma(E), \mathbf{c}(\sigma[E]))$ is homeomorphic to $(0,1)$ whenever $E \in \mathcal{E}_{\text{bdry}}$ there would be for each $i = 1, 2$,

$$E_i, \quad a_i, \quad b_i, \quad A_i, \quad B_i$$

such that

(i) $E_i \in \mathcal{E}_{\text{bdry}}$;
(ii) $a_i = \gamma(E_i)$ and $b_i = \gamma(\sigma[E_i])$;
(iii) A_i is the interior of $\mathbf{a}(a_i, b_i)$ relative to $\partial\Omega$;
(iv) B_i is the interior of the triangle which is the convex hull of $E_i \cup \sigma[E_i]$

but such that

$$B_1 \cap B_2 = \emptyset \quad \text{and} \quad \{a_2, b_2\} \cap A_1 \ne \emptyset.$$

Suppose $c \in \{a_2, b_2\}$ and $c \in A_1$. Since $B_1 \cap B_2 = \emptyset$, c would lie on an edge F of the triangle which is the closure of B_1. If c lay on the interior of F we would have $F = E_2$ or $F = \sigma[E_2]$ so that $F \in \mathcal{E}_{\text{bdry}}$, which would imply that $F = E_1$ or $F = \sigma[E_1]$; but then

$$c = \gamma(F) \in \{\gamma(E_1), \gamma(\sigma[E_1])\} = \{a_1, b_1\};$$

that is, $c \notin A_1$. So c is a vertex of F and therefore a vertex of the triangle which is the convex hull of $E_1 \cup \sigma[E_1]$. But this contradicts Lemma 8.2.

That $\mathbf{l}(\gamma)$ does not exceed L follows from the triangle inequality. □

Theorem 8.5. Suppose $\gamma \in \Gamma$. Then $\xi[\mathbf{p}(\gamma)] = \partial\Omega$ and

$$L \le \frac{R}{R-h}\mathbf{l}(\gamma)$$

where L is the length of $\partial\Omega$.

Proof. For each $E \in \mathcal{E}_{\text{bdry}}$ choose $\beta(E) \in E \cap \partial\Omega$.

Suppose $b \in \partial\Omega$ and let $B = \{x \in \mathbb{R}^2 : |(x-b)\bullet T(b)| < R \text{ and } |(x-b)\bullet N(b)| \le R\}$. Let $T_\pm \in \mathcal{T}$ be such that $b \pm RN(b) \in T_\pm$. Choose $v_\pm \in B \cap \mathbf{v}(T_\pm)$; this is possible since $h < R$. Choose a continuous map $\zeta : [0,1] \to B \cap (\cup\mathcal{E})$ such that

$\zeta(0) = v_- \in \Omega$ and $\zeta(1) = v_+$; this is possible since $h < R$. It follows from Theorem 7.1 that $K = \{t \in [0,1] : \zeta(t) \in \partial\Omega\}$ is a nonempty compact subset of $(0,1)$. Let $E \in \mathcal{E}$ be such that $\zeta(\inf K) \in E$. It follows that $E \in \mathcal{E}_{\mathrm{bdry}}$.

Given $\zeta \in \Gamma$ we can define a map $f_\zeta : Z_{1/2} \to \mathbf{p}(\zeta)$ by assigning $(1-t)\zeta(E) + t\zeta(\sigma[E])$ to $(1-t)\mu_{1/2}(E) + t\mu_{1/2}(\sigma[E])$ whenever $E \in \mathcal{E}_{\mathrm{bdry}}$ and $t \in [0,1]$.

Let C be the connected component of $\beta(E)$ in $Z_{1/2}$ and let D be the connected component of b in $\partial\Omega$. Let $\mu : \mathbb{S}^1 \to C$ and $\nu : \mathbb{S}^1 \to D$ be homeomorphisms. It follows from the preceding Lemma that the degree of $\nu^{-1} \circ \xi \circ f_\beta \circ \mu$ is ± 1. Since f_γ is homotopic to f_β we infer that the degree of $\nu^{-1} \circ \xi \circ f_\gamma \circ \mu$ is ± 1 which in turn implies that $D \subset \xi[\mathbf{p}(\gamma)]$.

Thus $\partial\Omega \subset \xi[\mathbf{p}(\beta)]$; this implies

$$L \le \mathbf{Lip}\,\xi \mathbf{l}(\beta) \le \frac{R}{R-h}\mathbf{l}(\beta).$$

□

8.3. **Proof of Theorem 8.1.** For each $E \in \mathcal{E}_{\mathrm{bdry}}$ choose $\beta(E) \in \partial\Omega \cap E$. From the preceding Theorem and ?? we infer that

$$\frac{R-h}{R}L \le \mathbf{l}(\gamma) \le L \le \mathbf{l}(\beta) \le L.$$

9. The tangent estimate.

We suppose throughout this section that $\gamma \in \Gamma_{\min}$.

Theorem 9.1. Suppose $E \in \mathcal{E}_{\mathrm{bdry}}$, $u \in \mathbb{S}^1$ is such that

$$\gamma(\sigma[E]) - \gamma(E) = |\gamma(\sigma[E]) - \gamma(E)|u,$$

and $a \in \mathbf{c}(\gamma(E), \gamma(\sigma[E]))$. Then

$$|u - T(\xi(a))| \le$$

Proof. We may assume without loss of generality that a is the midpoint of E, $\xi(a) = 0$, $T(0) = -\mathbf{e}_1$ and $N(0) = \mathbf{e}_2$. By ?? we obtain $f : (-R,R) \to (-R,r)$ such that $\partial\Omega \cap \mathbf{D}(0,\mathbf{e}_2,R) = f$.

For $0 < r < R$ let $S(r) = \{(x,y) \in \mathbb{R}^2 : |x| < r\}$. Since $E \cap \partial\Omega \neq \emptyset$,

$$|a| = \mathbf{dist}\, a, \partial\Omega \le \mathbf{diam}\, E < h.$$

In particular, $E \subset S(R-h) \cap \mathbf{D}(0,\mathbf{e}_2,R)$.

Let C be the connected component of a in $\mathbf{p}(\gamma) \cap S(R-h)$.

For each $i \in \mathbb{Z}$ let $E_i = \sigma^i[E]$; let $T_i = \mathbf{c}(E_i \cup E_{i+1})$; let $g_i = \gamma(E_i)$; and let $S_i = \mathbf{c}(b_i, b_{i+1})$.

Let β be a choice function on $\mathcal{E}_{\mathrm{bdry}}$ such that $\beta(E) \in \partial\Omega$ whenevr $E \in \mathcal{E}_{\mathrm{bdry}}$ and let $x_i = \xi(\beta(E_i)) \bullet \mathbf{e}_1$ for $i \in \mathcal{Z}$.

Let $\mathcal{F}$ be the largest connected subset of $\{F \in \mathcal{E}_{\mathrm{bdry}} : F \subset S(R)\}$. Since $E \in \mathcal{F}$ and since the diameter of the connected component of 0 in $\partial\Omega$ is at least $2\pi R$ we find that there are $I, J \in \mathbb{Z}$ such that $I \le 0 \le J$ and $\mathcal{F} = \{E_i : i \in \mathbb{I}(I,J)\}$. It follows that $(E_{I-1} \cup E_{J+1}) \subset \mathbb{R}^2 \sim S(R-h)$. Since the degree of the restriction of ξ to each connected component of $\mathbf{p}(\beta)$ is one we have

$$x_j \le x_i \quad \text{if } i,j \in \{I,J\} \text{ and } i < j.$$

It follows that that there are points $q_\pm$ such that $q_+ \in S_{I-1}$, $q_- \bullet \mathbf{e}_1 = R - h$, $q_+ \in S_J$ and $q_+ \bullet \mathbf{e}_1 = -R + h$. This in turn implies that

$$(b_{I-1} - q_-) \bullet \mathbf{e}_1 < 0 \quad \text{and that} \quad (q_+ - b_{J+1}) \bullet \mathbf{e}_1 < 0.$$

Let

$$\mathcal{F}_{\text{in}} = \{F\mathcal{E}_{\text{in}} : F \subset V_{\text{in}} \cap S\} \quad \text{and let} \quad \mathcal{F}_{\text{out}} = \{F\mathcal{E}_{\text{out}} : F \subset V_{\text{in}} \cap S\}$$

Suppose $\mathcal{G}$ is a nonempty maximal connected subset of $\mathcal{F}_{\text{in}}$. Let $I^* = \min\{i : E_i \in \mathcal{G}$ and let $J^* = \max\{i : E_i \in \mathcal{G}$.

Suppose $I < I*$ and $J* < J$. Then there are $p_+ \in E_{J^- -1} \sim E_{J^-}$ and $p_- \in E_{J^+}$ such that *either* $p_+ \bullet \mathbf{e}_1 = R - h$ *or* there is $x_\pm \in (-R+h, R-h)$ such that $p_\pm = (x_\pm, f(x_\pm))$ then

$$\mathbf{c}(g_{J^-}, p_-) \cap \mathbf{c}(g_{J^+}, p_+) \subset U^+.$$

Let

$$v_\pm = \frac{1}{\sqrt{1 + f'(x_\pm)^2}}(-f'(x_\pm), 1).$$

We have

$$u_{J^- -1} \bullet v_- \geq 0 \quad \text{as well as} \quad u_{J^+} \bullet v_+ \geq 0.$$

Let P be the union of the segments S_i, $i \in \{J^-, \ldots, J^- - 1\}$ and the segments $\mathbf{c}(p_-, g_{J^-})$ and $\mathbf{c}(g_{J^+ -1}, p_+)$ and let Q be the union of the segments $\mathbf{c}(p_\pm, (x_\pm, -R))$ and the segment $\mathbf{c}((x_+, -R), (x_+, -R))$. Then $P \cup Q$ is a simple closed polygon.

Let $\theta_i = \arcsin u_i \times u_{i+1}$ for $i \in \{J^-, \ldots, J^{-1}\}$; let $\alpha_- = \arcsin \mathbf{e}_2 \times u_{J^-}$ and let $\alpha_+ = -\arcsin u_{J^+} \times \mathbf{e}_2$. By the Gauss-Bonnet Theorem for simple closed polygons we infer that

$$\pi + \alpha_- + \alpha_+ + \sum_{i=J^-}^{J^+ -1} \theta_i = 2\pi.$$

Since $\theta_i \leq 0$ for $i \in \{J^-, \ldots, J^{-1}\}$ we find that $\alpha_- + \alpha_+ \geq \pi$. Since $|\alpha_\pm| \leq ??$ we find that $\alpha_\pm \geq 0$. Thus $P = \{(s, g(s)) : x_- \leq s \leq x_+\}$ for some convex $g : [x_-, x_+] \to [-R, R]$ for which $f(x_\pm) = g(x_\pm)$. Moreover, $f(x) \leq g(x)$ for $x \in [x_-, x_+]$ which implies that

$$f'(x_-) \leq g'(x_-) \quad \text{and} \quad g'(x_+) \leq f'(x_+).$$

Since g is convex we have

$$f'(x_-) \leq g'(x) \leq f'(x_+) \quad \text{whenever } x_- \leq x \leq x_+.$$

For each $\zeta \in \Gamma$ and $T \in \mathcal{T}_{\text{bdry}}$ let

$$\mathbf{q}_\pm(T, \zeta) \in \mathbf{H}$$

be defined as follows. Let E, F be such that $\mathcal{E}_{\text{bdry}} \cap \mathbf{e}(T) = \{E, F\}$ and $F = \sigma(E)$. In case $\zeta(E) \neq \zeta(F)$ we let

$$\mathbf{q}_\pm(T, \zeta) = T \cap \mathbf{int}\, \mathbf{h}_\pm(\gamma(E), \gamma(F)).$$

In case $\zeta(E) = \zeta(F) \in \mathcal{V}_{\text{in}}$ we let

$$\mathbf{q}_+(T, \zeta) = \emptyset \quad \text{and we let} \quad \mathbf{q}_-(T, \zeta) = T \sim \{\zeta(E)\}.$$

In case $\zeta(E) = \zeta(F) \in \mathcal{V}_{\text{in}}$ we let

$$\mathbf{q}_+(T, \zeta) = T \sim \{\zeta(E)\} \quad \text{and we let} \quad \mathbf{q}_-(T, \zeta) = \emptyset.$$

We let

$$U^+ = \cup_{i=I^-}^{J-1} \mathbf{q}_+(T_i, \beta) \sim \mathbf{cl}\, \mathbf{q}_+(T_i, \gamma)$$

and we let

$$U^- = \cup_{i=I^-}^{J-1} \mathbf{q}_+(T_i, \gamma) \sim \mathbf{cl}\, \mathbf{q}_+(T_i, \beta)$$

□

Proof.

$$b_i = \beta(E_i) \quad \text{and} \quad g_i = \gamma(E_i).$$

$$S_i = \mathbf{c}(b_i, b_{i+1}) \quad \text{and} \quad T_i = \mathbf{c}(g_i, g_{i+1}).$$

$$u, v : \mathcal{I} \to \mathbb{S}^1 \cup \{0\}$$

are such that

$$u_i = b_{i+1} - b_i \quad \text{and} \quad v_i = g_{i+1} - g_i.$$

Note that

$$b_i \bullet \mathbf{e}_1 < 0.$$

Let $\mathbb{M}_{\text{in}}$ be the set of $(A, B)\mathbb{I}(I, J) \times \mathbb{I}(I, J)$ such that $b_i \in \mathcal{V}_{\text{in}}$ whenever $i \in \mathbb{I}(A, B)$ and let $\mathbb{M}_{\text{out}}$ be the set of $(A, B)\mathbb{I}(I, J) \times \mathbb{I}(I, J)$ such that $\mathcal{V}_{\text{out}}$ whenever $i \in \mathbb{I}(A, B)$.

Lemma 9.1. $v_i \bullet \mathbf{e}_2 \neq 0$.

Proof. Suppose $v_i = s\mathbf{e}_2$ and $s \neq 0$. Then $S_i \subset \mathbf{h}_+(g_i, g_{i+1})$.
Suppose $s > 0$.

9.1. **Computing X.** For $0 < x < R$ let

$$f(x) = R - \sqrt{R^2 - x^2} \quad \text{and let} \quad g(x) = \frac{f(x) + h}{x}.$$

Then g has a unique minimum on $(0, R)$ at

$$X = \frac{R}{R + h}\sqrt{h^2 + 2Rh}$$

and both $f(X)$ and $f'(X)$ equal

$$\sqrt{\frac{h}{R}\left(\frac{h}{R} + 2\right)}.$$

Lemma 9.2. There are $I, J \in \mathbb{Z}$ such that $I < 0 < J$ and

$$\{i \in \mathbb{Z} : E_i \cap f \neq \emptyset\} = \mathbb{I}(I, J).$$

References

[AW2] W. K. Allard: *Total variation regularization for image denoising; I. Geometric theory.* to appear in SIAM Journal on Mathematical Analysis.

[FE] H. Federer: *Curvature Measures*, Transactions of the American Mathematical Society, Vol. 93 (Dec., 1959), pp. 418-491.