1 Tom's favorite integral

$$I = \int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}$$

One quick and dirty way to get it:

$$I = \int_0^\infty \frac{x^3 e^{-x}}{1 - e^{-x}}\,dx$$

Use the geometric series for $1/(1-z) = 1 + z + z^2 + \cdots$,

$$= \int_0^\infty x^3 e^{-x}\left[1 + e^{-x} + e^{-2x} + e^{-3x} + \cdots\right]\,dx$$

Switch the sum and integrals,

$$= \sum_{n=1}^\infty \int_0^\infty x^3 e^{-nx}\,dx$$

Use Laplace transform integral result: $\int_0^\infty t^3 e^{-st}\,dt = 3!/s^4$,

$$= \sum_{n=1}^\infty \frac{6}{n^4}$$

Use result from Fourier series, $\sum_{n=1}^\infty n^{-4} = \pi^4/90$,

$$I = 6 \cdot \frac{\pi^4}{90} = \frac{\pi^4}{15}.$$

Also fun to do via complex contour integration...or other methods??