Math 577: Mathematical Modeling Spring 2024

Problem Set 9 **Assigned Fri Apr 5** **Due Sat Apr 13**

Wave eqns and the method of characteristics

-1. $\boxed{\textbf{Test\#2: date TBD}}$ covering the material on HW's 5-9 and corresponding sections from Logan and class lectures (ending with Lecture 23): weakly nonlinear oscillators, Poincare-Lindstedt, multiple timescales, phase plane analysis, calculus of variations, and transport models/wave equations.

0. Reading: Logan, sections 7.1.1, 7.2.1 and for more background 6.2.1-2, 8.1.

1. Find the explicit solution for $\rho(x,t)$ for $t \geq 0$ for the initial value problem:

$$\frac{\partial \rho}{\partial t} + (2x + 4t - 2)\frac{\partial \rho}{\partial x} = 3\rho + 9x \qquad \text{with } \rho(x,0) = 5x^2 + 2.$$

First obtain the solution in parametric form, $\{X(t,A), P(t,A)\}$ then reduce that to the explicit solution for $\rho = \rho(x,t)$. (Hint: Review LCC equations and undetermined coefficients.)

2. An isolated one-dimensional compressible fluid blob starts at $t = 0$ with unit length and uniform density $\rho \equiv 1$ on $1 \leq x \leq 2$. The blob obeys the conservation of mass equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0.$$

where the (Eulerian) velocity field given as $v(x,t) = e^{2t}/x$. Find the density in blob for $t \geq 0$ as a function of position and time, $\rho = \rho(x,t)$ and the positions of the left and right edges of the moving blob, $x_1(t)$ and $x_2(t)$.[1]

3. <u>Types of starting data</u>[2]: Consider the PDE for $\rho(x,t)$ for $t \geq 0$: $\quad \dfrac{\partial \rho}{\partial t} + (2t+2)\dfrac{\partial \rho}{\partial x} = 3x$

   (a) Obtain the general solution in parametric form, $X(t), P(t)$. There will be two constants of integration – leave them general.

   (b) Draw the family of $X(t)$ characteristic curves in the (x,t) plane for $t \geq 0$.

   (c) Find the solution $\rho(x,t)$ for all x and $t \geq 0$ for the <u>*initial value problem*</u> with data given for all x at $t = 0$: $\rho(x,0) = 7\sin(x)$.

   (d) Find the solution $\rho(x,t)$ for the <u>*signaling problem*</u> where data is given at a fixed position ($x = 0$) over a range of times: for $t \geq 0$, $\rho(0,t) = 3\cos(2t)$.
   Hint: Here, you need to parametrize the data and characteristics differently since you are not given initial conditions for a range of x at a fixed time. You might parametrize by the time $t = B \geq 0$ with $X(B) = 0$ for each characteristic and $P(B)$ equals each starting value. Then apply algebra to eliminate the parameter B to get $\rho(x,t)$.

4. (2018) The solution $u(x,y)$ in the first quadrant of the xy plane ($x \geq 0$ and $y \geq 0$) obeys the semi-linear partial differential equation: $\quad (2x+2)\dfrac{\partial u}{\partial x} + (y-5)\dfrac{\partial u}{\partial y} = (3y - 15)u^2$

   (a) Write the characteristic ODE's resulting from assuming $U_a(x) = u(x, Y(x))$. *(continued)*

[1] From conservation of mass it should be true that $\int_{x_1(t)}^{x_2(t)} \rho(x,t)\,dx = 1$ for all times $t \geq 0$. You can let Maple or Mathematica check this...

[2] Consider initial data giving a 'snapshot' of the solution everywhere at the initial time, boundary data describing varying light levels coming from a lighthouse as viewed from positions on the ocean, and moving data as describing the sound being heard from different positions due to a siren from a moving car.

(b) Write the characteristic ODE's resulting from assuming $U_b(y) = u(X(y), y)$.

(c) Write the solution $u(x, y)$ and sketch/describe the domain of definition: for boundary conditions given by:

   i. $u(x, 0) = p(x)$ given on the x-axis for $x \geq 0$.

   ii. $u(0, y) = q(y)$ given on the y-axis for $y \geq 0$.

5. (2020) Consider the partial differential equation for $\rho(x, t)$ for $t \geq 0$:

$$\frac{\partial \rho}{\partial t} + \left(\frac{x}{t+1} + t + 1\right)\frac{\partial \rho}{\partial x} = 1.$$

(a) Obtain the general solution in parametric form, $X(t), P(t)$ with two constants of integration (call them C_1, C_2).
Hint: Use the trial solution $X(t) = a_2t^2 + a_1t + a_0$, find the a_k's.

(b) The values of the solution are given on a line segment as

$$\rho = f(x) \qquad \text{on } t = -x + 3 \text{ for } 0 \leq x \leq 2.$$

   i. Obtain the parametric equations for the solution.

   ii. Determine the range $a \leq x \leq b$ where ρ can be determined on the x-axis ($t = 0$) from this data.

   iii. Determine the explicit solution on the x-axis, $\rho(x, 0)$, in terms of the fcn f.

   Hint: You can get $\rho(x, 0)$ without calculating the messier $\rho(x, t)$.
   Hint: Have you checked your choice of $\pm$ sign?

6. Consider the system of wave equations

$$\begin{aligned} p_t + 5p_x - 7q_x &= 0 \\ q_t + 2p_x - 4q_x &= 0 \end{aligned}$$

(a) Find the eigenvalues and eigenvectors for traveling waves in this system.
In which directions do the two waves move?

(b) Find the solutions $p(x, t)$ and $q(x, t)$ if the initial conditions at $t = 0$ are

$$p(x, 0) = 5x \qquad q(x, 0) = 3x^2$$

7. Consider the inviscid Burgers' equation:

$$\frac{\partial \rho}{\partial t} + 2\rho\frac{\partial \rho}{\partial x} = 0,$$

starting with the "chopped parabola" initial conditions $\rho(x, t = 0) = \max(\,\frac{1}{4}x(6 - x)\,, 0)$.

(a) Use the method of characteristics to write the parametric solution.

(b) Eliminate the parameter from (a) to obtain a multi-valued solution in two nontrivial parts.
State which $\pm\sqrt{\ }$ corresponds to which part of the solution. (Hint: plot $\rho_\pm(x)$ at some fixed times?)

(c) Determine the time t_* and x_*-position where $X(t)$ characteristic curves first cross.

(d) Use (b,c) with the shock speed equation to write the ODE and IC for the initial value problem for the shock position $s(t)$. (DO NOT try to solve this eqn.)

(e) Write Burgers' equation as a conservation law (in divergence form), state the conserved quantity (with the specific value for this IC), and integrate using the solution from (b) and simplify to produce an algebraic equation involving t and $s(t)$. (DO NOT try to solve this eqn.)