Math 577: Mathematical Modeling — Spring 2024

Problem Set 8 — **Assigned Fri Mar 22** — **Due Monday, Apr 1**

Intermediate Calculus of Variations

0. Reading: Logan Chapter 4, Sections: 4.4 and 4.6 and Lectures 18-20. Questions 1-4 can be done using information from Logan and the lectures-to-date. Questions 5 and 6 on optimal control will be covered next week in Lecture 20 (Logan does not do this topic); I recommend waiting until after Lecture 20 to do those problems (note the extended due date for this assignment).

1. Logan (Exercises for Sect 4.4) page 253, Problems 5d and 5e ("Find the extremals for ... $y(1)$ free."). (page 185 in the 3rd edition)

2. Obtain the function $y = y_*(x)$ on $0 \le x \le b_*$ that connects the origin to the curve $y = 1 + (x-1)^2$ at $x = b_*$ and minimizes the functional
$$J(y) = \frac{1}{2}\int_0^b \left(\frac{dy}{dx}\right)^2 dx.$$
Your final answer should be a specific function for $y_*(x)$ and a value for $x = b_*$.

3. (2021) Find a locally optimum solution $y(x)$ on $0 \le x \le 1$ of the functional
$$J = \int_0^1 \left[x\,\frac{dy}{dx}\,\frac{d^2y}{dx^2}\right] dx - \frac{1}{2}\left(\left.\frac{dy}{dx}\right|_{x=1}\right)^2$$
subject to the constraint
$$\int_0^1 y\,dx = \frac{1}{2}\,y(0)^2 + \frac{1}{2}\,y(1)^2\ .$$
   (a) Determine the second-order ODE for $y(x)$ that is obtained from the first variation by assuming that the boundary terms vanish.
   (b) Determine the two necessary boundary conditions that are needed to make all the boundary terms vanish.
   (c) Solve the ODE problem to determine the optimal solution $y(x)$.

4. <u>THE classic isoperimetric problem</u>: This problem will lead you through the steps deriving the result that the circle is the smooth closed curve of a fixed perimeter that encloses the maximum area.
Let $x(t), y(t)$ be the parametric equations for a closed curve on $0 \le t \le 1$ that goes through the origin:
$$x(0) = x(1) = 0 \qquad y(0) = y(1) = 0$$
   (a) Use Green's theorem from multi-variable calculus to relate the area enclosed by the closed curve to the line integral on the curve:
$$\int_0^1 \left[x(t)y'(t) - y(t)x'(t)\right] dt$$
   (b) Consider the augmented objective function
$$\mathcal{L}(x, x', y, y', \lambda) = (xy' - yx') - \lambda\left[\sqrt{(x')^2 + (y')^2} - P\right] \qquad J(x, y, \lambda) = \int_0^1 \mathcal{L}\,dt$$
   where P is the perimeter. Determine the Euler-Lagrange equations.
   (c) Integrate each EL once with respect to t and show that they can be combined to yield the equation for a circle having $|\lambda|$ related to the radius.
   (d) Determine λ so that the perimeter constraint is satisfied. Determine the possible positions for the center of the circle.

(continued)

5. The Pontryagin Maximum Principle (PMP) is an alternative (shortcut) approach to writing the equations for classic/basic optimal control problems with free-end time in terms of a Hamiltonian, $H(x, u, \lambda) = L(x, u) + \lambda f(x, u)$, instead of using the Euler-Lagrange equations:[1]

   (a) Show that the state equation ODE can be written as "the rate of change of the state equals the derivative of H with respect to the co-state".

   (b) Show that the co-state equation ODE can be written as "the rate of change of the co-state equals minus the derivative of H with respect to the state".

   (c) Show that the control equation can be written as "the derivative of H with respect to the control is zero".

   (d) Use the chain rule to show that $H(x(t), u(t), \lambda(t))$ is a constant for an optimal solution.

   The "M" in PMP comes from the fact that it can be shown for all trial solutions, $\tilde{H} \le H_* = 0$ when the end-time T is left free. Like the EL, the PMP also generalizes to give systems of equations for problems where the states/co-states are vectors like $\mathbf{x}(t) = (x_1(t), x_2(t), \cdots)$ and $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \cdots)$.

6. Determine the solution $x(t)$ and the control function $u(t)$ that satisfy the state equation

$$\frac{dx}{dt} = 3x + u \qquad 0 \le t \le T$$

   with initial and final conditions

$$x(0) = 2 \qquad x(T) = 1$$

   while minimizing the cost functional

$$J = \int_0^T \left(4x^2 + 3xu + u^2\right)\, dt$$

   (a) Use the Pontryagin principle for the case where the final time is the optimal stopping time $T = T_*$.

   (b) If instead of the optimal stopping time, if the final time is specified as $T = 1/4$, what is the optimal solution? What is the value of the Hamiltonian?

   Hint: Nothing changes in the derivation from the lecture except for things related to T_* and H.

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[1] The Euler-Lagrange ODE $L_x - \frac{d}{dt}(L_{x'}) = 0$ is the shortcut for Lagrangian mechanics with $L = L(t, x(t), x'(t)) = T - V$. There is also a similar Hamiltonian mechanics shortcut from the constant total energy, $H = H(x(t), x'(t)) = T + V$; it gets written as two first-order equations: $\frac{dx}{dt} = H_{x'}$ and $\frac{dx'}{dt} = -H_x$, see Logan Section 4.5.1 for more information. The PMP is closely related.