Math 577: Mathematical Modeling — Spring 2024

Problem Set 7 — **Assigned Fri Mar 8** — **Due** $\boxed{\textbf{Sat Mar 23}}$

Introduction to the Calculus of Variations

0. Reading: Logan Chapter 4, sections: 4.1, 4.2.2, 4.3, 4.5.1, 4.4.

1. Use the classic Euler-Lagrange eqn to solve problems from Logan, page 243 (page 175 in 3rd Edition):

   (a) Problem 2a with $a = 1$ and $b = 2$ and boundary conditions $y(1) = 1$ and $y(2) = 2$.

   (b) Problem 4.

2. **The second derivative test**: Recall the problem of minimizing the arclength of the curve $y(x)$ from $(0, 0)$ to $(1, b)$:

$$\min_y J = \int_0^1 \sqrt{1 + (y')^2}\, dx \qquad y(x) = y_*(x) + \epsilon h(x)$$

   In class we expanded J to $O(\epsilon)$ to obtain the critical point condition leading to the Euler-Lagrange equation for $y_*(x)$.

   (a) Expand J to $O(\epsilon^2)$ to obtain the second variation, $\delta^2 J$, this is an integral that can have combinations of $h(x)$ and $h'(x)$ in it (do not try to "reduce" it further).

   (b) Show that the solution $y_*(x)$ is indeed a local minimum by showing that the second variation is positive for all nontrivial $h(x)$.

   For this problem, it is possible to show that any critical point could not be a maxima (without doing any calculations): draw any $\tilde{y}(x)$ – it is always possible to make it even longer... so no max.

3. The Euler-Lagrange equations for constrained motion can be obtained by starting from the action integral with the Lagrangian for general unconstrained motion and then plugging-in the parametric equations describing the geometric constraint (the equations for the curve or surface) before carrying out the variation with respect to the remaining variables.[1]

   (a) Consider the action integral for two-dimensional motion of a mass subject to gravity

$$I = \int \tfrac{1}{2} m \left[x'(t)^2 + y'(t)^2\right] - mgy(t)\, dt$$

   Consider the mass to be constrained to be on a circle, $x^2 + y^2 = \ell^2$. Derive the equation of a pendulum by first plugging the parametric equations $x(t) = \ell \cos\theta(t)$, $y(t) = \ell \sin\theta(t)$ into I and then applying the principle of least action for $I(\theta(t))$.

   (b) Consider the action integral for three-dimensional motion with gravity

$$I = \int \tfrac{1}{2} m \left[x'(t)^2 + y'(t)^2 + z'(t)^2\right] - mgz(t)\, dt$$

   Derive the equations of motion for a ball rolling on the surface of a cone, $x^2 + y^2 = z^2$, by first plugging the parametric equations $x(t) = r(t)\cos\theta(t)$, $y(t) = r(t)\sin\theta(t)$, $z(t) = r(t)$ into I and then applying the principle of least action for $I(r(t), \theta(t))$.

4. (2019) Consider the problem of finding a locally optimum solution of the functional

$$J = \int_0^1 \left[2y\frac{d^2 y}{dx^2} + \int_0^{y(x)} e^{-t^2}\, dt + 3x^2 \int_x^1 [y'(t)]^2\, dt \right] dx$$

   subject to the boundary condition $y'(0) = y(0)$. Express the first variation of J in terms of an integral, $\int_0^1 [\text{ODE for } y(x)] h(x)\, dx$, and boundary terms. *(continued)*

[1] This approach will be much shorter than the method using Lagrange multipliers that we will learn later. The Lagrange multiplier approach is needed when parametric equations for the surface cannot be used or are not available.

5. **NBC's for the beam equation**: The Lagrangian ($L = T - V$) and Hamilton's principle of least action ($J = \int L\,dt$ and $\delta J = 0$) also be applied to determine time-dependent partial differential equations. Let $y = u(x,t)$ be the transverse deflections of a rod or beam with length ℓ (domain: $0 \le x \le \ell$) with constant mass density ρ (mass per unit length), constant bending stiffness E (like a spring constant) and constant moment of inertia I.
The overall total kinetic and potential energies of the beam are given by integrals over the length of the beam of the corresponding energy density functions, $e(u)$, $\mathcal{E} = \int_0^\ell e(u)\,dx$. The kinetic energy density is $\frac{1}{2}\rho(\partial_t u)^2$. The potential energy density due to bending (i.e. transverse deflection or buckling) is $\frac{1}{2}EI(\partial_{xx}u)^2$. So the action integral for this problem is

$$J(u) = \int_{t_0}^{t_1}\int_0^\ell \left[\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2\right] dx\,dt$$

Consider trial solutions $\tilde{u}(x,t) = u_*(x,t) + \epsilon h(x,t)$.

Apply the principle of least action to derive the beam equation for $u(x,t)$ and state all **four** sets of the combinations of the two natural boundary conditions **at** $x = 0$ on u (or $u_x, u_{xx}, \ldots$) [no h's or ϵ's] for this problem. Each BC will be a single-term homogeneous or inhomogeneous BC.[2]

Hints:

- $J = \int_{t_0}^{t_1} L\,dt$. J is a double integral. Interchange order of integration when it might help. Assume the variation of the solution is zero everywhere in x along the beam at start/end times, $t = t_0$ and $t = t_1$.
- Assume the boundary conditions at $x = 0$ and $x = \ell$ are separated – there is no coupling between conditions at $x = 0$ and conditions at $x = \ell$. All of the terms from $x = \ell$ can be handled similarly to your NBC's for $x = 0$.
- The Lagrangian contains second derivatives. This just means that you'll need to do more integration-by-parts to get all of the derivatives off of $h(x,t)$, but otherwise everything works as usual.

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6. (Optional, extra credit) **The brachistochrone problem** (meaning *"least time"* in Greek): Consider the problem of finding the curve $y = y(x)$ (shape of a ramp or slide[3]) for least-time descent for a sliding object under the influence of gravity (acting in the y direction) starting from rest at from $(x,y) = (0,1)$ at time $t = 0$, and ending at $(x,y) = (1,0)$ at $t = T$.
The integral for the time of travel can be expressed in terms of the speed of the object and shape of the ramp by

$$T(y) = \int_0^T dt = \int \frac{ds}{ds/dt} = \int_0^1 \frac{\sqrt{1+y'(x)^2}}{v(y)}\,dx$$

where the formula for arclength ($ds = \sqrt{1+y'(x)^2}\,dx$) and the relation between the speed and the arclength (the distance traveled) ($v = ds/dt$) were used.
Use the following steps to solve the problem:

(a) Assume there is no friction, so the total energy is $E = \frac{1}{2}mv^2 + mgy$, and it remains constant, set by its initial value at $t = 0$. Use this to write an expression for $v(y)$.

(b) Write the Euler-Lagrange ODE for $y_*(x)$ for the functional $T(y)$.

(c) Noting that the integrand function does not explicitly depend on the independent variable x, use the Beltrami identity to obtain a simpler first-order ODE with for $y_*(x)$ with an extra constant, call it C.

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[2] Assume the boundary conditions are separated – there is no coupling between conditions at $x = 0$ and conditions at $x = \ell$. The two BC's at $x = \ell$ will then have the same kinds of options.

[3] Imagine a part of a roller coaster ride