Math 577: Mathematical Modeling — Spring 2024

Problem Set 5 — **Assigned Fri Feb 16** — **Due Sat Feb 24**

Perturbation methods for weakly-nonlinear oscillator problems

-1. **Math 577 Test 1** **Weds, Feb 28th, in class, on paper.** Covers the course so far: HWs 1-4, Lectures 1-9, Logan, Chaps 1,3: scaling and nondimensionalization, similarity solutions, perturbation problems and ODE boundary layers [NO oscillator problems].

For the week of the test there will be no HW due, just study/practice and prepare for the test! Office hours will be shifted appropriately (new when2meet)

0. **Reading: Logan, sections 3.1.2 and 3.1.3 and lecture notes (10, 11).**

1. For each problem use the Poincare-Lindstedt method to determine the two-term approximation of the solution, $x(t) \sim \tilde{x}_0(\tau) + \epsilon \tilde{x}_1(\tau)$ with $\tau = (\omega_0 + \epsilon\omega_1)t$. Find $\omega_0, \tilde{x}_0(\tau), \omega_1$ and $\tilde{x}_1(\tau)$ (**in that order**) for:

   (a)
   $$\frac{d^2x}{dt^2} + 4x = -\epsilon x^3 \qquad x(0) = a \qquad x'(0) = 0$$
   This problem (called the Duffing oscillator) has a conserved quantity for each solution (called the Hamiltonian), corresponding to the total mechanical energy. This problem has periodic oscillations for any amplitude $a > 0$.

   (b)
   $$\frac{d^2x}{dt^2} + 9x = -\epsilon(x^2 - 1)\frac{dx}{dt} \qquad x(0) = a \qquad x'(0) = 0$$
   This problem (called the van der Pol oscillator) has nonlinear damping/driving. Most of its solutions have growing or decaying amplitudes. But there is one special IC value $a_* > 0$[1] yielding a periodic solution (called the limit cycle oscillation [LCO]). Find this a_*.

2. Use the method of multiple scales with $T = \epsilon t$ for the van der Pol oscillator
   $$\frac{d^2x}{dt^2} + 9x = -\epsilon(x^2 - 1)\frac{dx}{dt} \qquad \epsilon \to 0$$
   to obtain a solution in the form $x(t) \sim \tilde{x}_0(t, T) = A(T)\sin(3t) + B(T)\cos(3t)$.

   (a) Determine the amplitude equations for $A(T), B(T)$.

   (b) Let $R(T) = \sqrt{A^2 + B^2}$. Use the amplitude equations for $A(T), B(T)$ to determine the equation for $dR/dT = f(R)$. Determine the equilibrium (steady-state) values for R.

3. (Computer-aided algebra recommended) Use the Poincare-Lindstedt method for the problem
   $$\frac{d^2x}{dt^2} + 25x = 12\epsilon\left(\frac{dx}{dt}\right)^2 \qquad x(0) = 1, \qquad x'(0) = 0,$$
   to find the first nontrivial correction to the oscillation frequency. How many terms in the expansion of $x(t)$ have you determined in obtaining that correction?

(continued)

[1] $a = 0$ yields the trivial solution, $x(t) \equiv 0$, not helpful.

4. Polar form and Fourier series: Use the method of multiple scales with $T = \epsilon t$ for

$$\frac{d^2x}{dt^2} + \epsilon \left|\frac{dx}{dt}\right| \frac{dx}{dt} + x = 0, \qquad x(0) = 0, \qquad x'(0) = 1, \qquad \epsilon \to 0.$$

(a) Show that the leading order solution can be written in the polar form: $\tilde{x}_0(t,T) = R(T)\sin(t+\Phi(T))$. Relate the amplitude R and phase Φ to the coefficients A, B in $\tilde{x}_0 = A\sin(t) + B\cos(t)$. What are the initial conditions for R, Φ?

(b) Derive and solve the amplitude equations for $R(T)$ and $\Phi(T)$ to obtain the leading order solution $x(t) \sim \tilde{x}_0(t,T)$.
Hint: You will need to calculate some terms in the Fourier series of the RHS forcing. Write the series in terms of the variable $s = t + \Phi$ on $-\pi < s < \pi$, namely $\sum_{k=0}^{\infty} a_k \sin(ks) + b_k \cos(ks)$. (How many terms do you really need?)

5. A damped, driven Duffing oscillator near resonance and $e^{\pm it}$: For $\epsilon \to 0$, consider the problem for $x(t)$,

$$\frac{d^2x}{dt^2} + \epsilon\beta\frac{dx}{dt} + x + \epsilon\alpha x^3 = \epsilon\cos(t + \gamma\epsilon t),$$

with given parameters α, β, γ. Use the slow-timescale $T = \epsilon t$ in the method of multiple scales. Note the presence of τ in the forcing term on the RHS. Hint: $\cos(t + \gamma T)$.

(a) Show that the leading order solution can be written in the complex form $\tilde{x}_0(t,T) = C(T)e^{it} + \overline{C(T)}e^{-it}$, where $\overline{z} = x - iy$ is the complex conjugate of $z = x + iy$. Express the complex-valued function $C(T)$ in terms of the real-valued functions $A(T), B(T)$ used in $\tilde{x}_0 = A(T)\sin t + B(T)\cos t$.

(b) Using (a) in the equation for $\tilde{x}_1(t,T)$ find the two solvability conditions. Show that these reduce to a single complex equation for dC/dT.
Hint: This is easier with complex exponentials, $e^{\pm it}$, rather than trig fcns.

(c) The phenomenon of *entrainment* describes a periodic solution locking onto the behavior entirely set by a forcing term, leaving no sign of the influence of the natural frequency from the unforced problem (i.e. no homogeneous solution terms).
Setting $C(T) = Me^{i\theta}e^{i\gamma T}$ in your equation from (b) where M is a (real-valued) constant magnitude and θ is a (real-valued) constant phase. Find a formula for γ in terms of M, $\gamma = \gamma(M)$, this is sometimes called a detuning relation. Note that $\alpha, \beta, \gamma, \theta, M$ are real-valued constants. Separate your result into real/imaginary parts to obtain two equations for γ, θ. (Do not try to solve these equations)

6. (2020) Consider the perturbed fourth-order oscillator equation for $x(t)$:

$$\frac{d^4x}{dt^4} + 10\,\frac{d^2x}{dt^2} + 9\,x = 2\epsilon\, x\,\cos(2t).$$

Using the method of multiple time scales, the solution can be expressed as $x(t) = \tilde{x}(t,T)$ with $T = \epsilon t$.

(a) Use the chain rule to write the full partial differential equation for $\tilde{x}(t,T)$ with $\epsilon > 0$, where t, T are considered as independent timescales. Hint: Use $(d/dt)^n = (\partial_t + \epsilon\partial_T)^n$ for any $n = 1, 2, \cdots$.

(b) For $\epsilon \to 0$, the solution can be written as an expansion, $\tilde{x} \sim \tilde{x}_0(t,T) + \epsilon\tilde{x}_1(t,T)$. Write the general form of the leading order term as the sum of four trigonometric terms with coefficients $A(T), B(T), C(T), D(T)$.
Hint: The characteristic polynomial can be factored as $(m^2 + \alpha)(m^2 + \beta)$.

(c) Write the amplitude equations that would need to be solved to determine A, B, C, D. (Do not try to solve these eqns!)