Math 577: Mathematical Modeling — Spring 2024

Problem Set 4 **Assigned Fri Feb 9** **Due Sat Feb 17**

ODE Boundary Layer Problems

-1. **Math 577 Test 1** : (Date to be decided), in-class, covering the course material so far (HW's 1–4, Lectures 1–9, Logan Chaps 1 and 3 (the parts we have covered only): scaling and nondimensionalization, similarity solutions, perturbation problems and ODE boundary layers [NO oscillator problems]). No books and no calculators allowed. You will be given the "basic math summary" review sheet and you can bring **one letter-size sheet of your own handwritten notes**.

Previous tests are posted on Canvas to guide your studying. I strongly encourage everyone to study with others in the class and exchange questions/progress via the Ed Discussion message board.

0. Reading: Logan, sections 3.2 and 3.3.
Also, note that the small parameter $\epsilon \to 0$ is always assumed to be positive, $\epsilon \geq 0$.

1. Consider the boundary value problem for $y(x)$ on $0 \leq x \leq 1$ with $\epsilon \to 0$:

$$\epsilon \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + e^{2y} = 0, \qquad y(0) = 5\,\epsilon, \qquad y(1) = -8\,\epsilon.$$

You can assume that the solution is $O(1)$ on the entire domain. (It is true!)[1]

(a) For this problem the logic that *"If there's a BL on one end, then the outer solution will get the other BC. Both BC's are $O(\epsilon)$ so that must pick $\beta_{\text{out}} = 1$."* does NOT work.
Try $y(x) \sim \epsilon \mathcal{Y}(x)$ and show it leads to a contradiction.
Try $y(x) \sim y_0(x) + \epsilon y_1(x)$ (i.e. $\beta_{\text{out}} = 0$) and write the equations for $O(\epsilon^0)$ and $O(\epsilon^1)$.

(b) Determine the two distinguished limits (α's) for this problem from the ODE.

(c) Determine the general solution for the leading order outer solution $y \sim y_0(x)$.

(d) Determine the leading order inner solution $y \sim Y_0(X)$ and determine where the boundary layer occurs.

(e) Write the leading order uniformly-valid solution.

(f) Use $y_0(x)$ from above to determine the next term in the expansion of the outer solution, $y_1(x)$.

2. (2021) Consider the ODE problem for $y(x)$ on $0 \leq x \leq 1$ in the limit $\epsilon \to 0^+$,

$$\epsilon^4 (1-x)\,\frac{d^3y}{dx^3} + \epsilon x\,\frac{d^2y}{dx^2} - 4\,\frac{dy}{dx} = 36x^2.$$

You are given that the solution is bounded on the whole domain, $y(x) = O(1)$.

(a) Find the first two terms in the expansion of the outer solution on $0 < x < 1$, $y \sim y_0(x) + \epsilon y_1(x)$.

(b) The solution has a boundary layer at $x_* = 0$. Determine the leading order inner solution that satisfies the boundary condition $y(0) = 5$.

(c) The solution has a boundary layer at $x_* = 1$. Determine the leading order inner solution that satisfies the boundary conditions $y(1) = -5$ and $y'(1) = 8/\epsilon$.

(d) Find the un-determined constants in the leading order solutions from parts (a,b) and write the uniform/composite leading order solution.

(continued)

[1] $y = O(1)$ means y is finite in magnitude as $\epsilon \to 0$, like $|y(x)| < M$, $O(1)$ also includes anything smaller.

3. Consider the boundary value problem for $y(x)$ on $0 \le x \le 1$ with $\epsilon \to 0$:

$$\epsilon \frac{d^2y}{dx^2} + (1-4x)\frac{dy}{dx} - y = 0 \qquad y(0) = -1 \qquad y(1) = 1$$

You can assume that the solution is $O(1)$ on the entire domain.

(a) Find the general solution for the leading order outer solution. Determine its constant of integration so that $y_0(x)$ is finite and real-valued everywhere in $0 < x < 1$.

(b) Determine the boundary layer solutions and write the composite leading order solution.

(c) If the sign of the first term in the ODE is changed, $-\epsilon \frac{d^2y}{dx^2} + (1-4x)\frac{dy}{dx} - y = 0$, show that the solution has an interior boundary layer at $x_* = 1/4$. Determine the scaling for this inner solution and write the ODE for $Y_0(X)$ (but do not solve it). Determine the constants of integration that would be needed for the outer solutions on $0 \le x < 1/4$ and $1/4 < x \le 1$ that satisfy the BC's.

4. (2020) Consider the ODE for $y(x)$ on $0 \le x \le 2$ in the limit $\epsilon \to 0^+$,

$$\epsilon^2 \frac{dy}{dx}\frac{d^2y}{dx^2} - \epsilon(3+x)\left(\frac{dy}{dx}\right)^2 + 5y^2\frac{dy}{dx} = 4\epsilon^2.$$

Consider the four possible dominant balances of three-terms for $\epsilon \to 0$ that can be obtained from this ODE using the scaling $y = \epsilon^\beta Y$, $X = x/\epsilon^\alpha$. Find $\{\alpha, \beta\}$ in each case **and** identify if each yields a valid distinguished limit.

5. (2022) Consider the ODE problem for $y(x)$ on $2 \le x \le 3$ in the limit $\epsilon \to 0^+$,

$$\epsilon(4-x^2)\frac{d^2y}{dx^2} + \epsilon^3(3-x)\left(\frac{dy}{dx}\right)^2 + 20y = 40x^2.$$

You are given that the solution is bounded on the whole domain, $y(x) = O(1)$.

(a) Find the first two terms in the expansion of the outer solution on $2 < x < 3$, $y \sim y_0(x) + \epsilon y_1(x)$.

(b) The solution has a boundary layer at $x_* = 3$. Determine the leading order inner solution that satisfies the boundary condition $y(3) = 7$.

(c) The left boundary condition on the solution is $y(2) = 5$.
The solution has a double (nested) boundary layer structure at $x_* = 2$ with

$$y = Y^A(X^A) \qquad X^A = \frac{x-2}{\epsilon^{\alpha_A}} \qquad \text{and} \qquad y = Y^B(X^B) \qquad X^B = \frac{x-2}{\epsilon^{\alpha_B}}$$

Determine α^A, α^B and the leading order equations for Y_0^A and Y_0^B. (DO NOT solve these ODE's)

(d) Determine c_1, c_2, c_3, c_4, c_5 in

$$5 = \lim_{X^A \to c_1} Y_0^A \qquad \lim_{X^A \to c_2} Y_0^A = \lim_{X^B \to c_3} Y_0^B \qquad \lim_{X^B \to c_4} Y_0^B = \lim_{x \to 2} y_0(x) = c_5$$

Hint: What is the difference between Y_0^A and Y_0^B?

The "nested" boundary layer represents an narrower inner solution within another (wider) inner solution, called inner-inner and inner solutions respectively (sometimes also called a 'triple deck' for outer, inner, and inner-inner solns). Constructing the overall solution would involve applying the BC to the inner-inner solution, matching the inner-inner to the inner (sometimes also called the "intermediate layer"), then matching the inner to the outer solution. Or, said differently – *"Relative to the outer soln's point of view, the inner solution lives at the boundary, the outer soln lives far away from the boundary. Relative to the inner soln, the inner-inner soln lives at the boundary, while I live far from the boundary. Relative to the inner-inner's POV, I have the BC, the inner lives far away. Outer soln? Never heard of it..."*