Math 577: Mathematical Modeling — Spring 2024

Problem Set 3 — **Assigned Fri Feb 2** — **Due Sat Feb 10**

Regular and Singular Perturbation Problems

-1. Computer-Assisted Algebra Policy: You may always use Maple or Mathematica (or other computer software) to help check your work on problem sets. For problems marked as "Computer-Algebra Recommended" (CAR), parts of the algebra may be somewhat long and tedious to work out by hand, so you can use computer-based assistance to carry out the work. For these problems, you must include copies of the PDF pages of the complete steps and computer-generated solutions. For other problems (Non-CAR), you should write out your solution process by hand.

0. Reading from Logan: Sections 3.1, 3.2.
Also see the `L05-demo.mws`, `L05-expansion.mws` and `L05-iteration.mws` (expansion and iteration methods for the projectile motion example) files posted on Canvas for a guide on how to do the algebra for perturbation expansions via `Maple`.

1. (Computer-Algebra Recommended) Consider the problem for the vertical motion of a projectile:

$$\frac{d^2x}{dt^2} = -\frac{1}{(1+\epsilon x)^2}\,, \qquad x(0) = 8\,, \qquad x'(0) = 9\epsilon \qquad \text{with } \epsilon \to 0.$$

(a) Obtain the first three terms in the expansion of the solution, $x \sim x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t)$.
Note: the initial height is $O(1)$ but the initial speed is slow, $O(\epsilon)$.

(b) Let T be the time when the projectile lands on the ground, $x(T) = 0$. Use (a) to determine $T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + O(\epsilon^3)$. Does it take more or less time to land compared to landing on a "flat Earth"?
Note: if you don't keep enough terms in the AE's for x and T your answers for T_1, T_2 may not be correct.

(c) The solution from Lecture 5[1] can be used with IC's $x(0) = \beta = O(1)$ and $x'(0) = \alpha = O(1)$ to yield

$$x^{\alpha\beta}(t) = \left(\beta + \alpha t - \frac{1}{2}t^2\right) + \epsilon\left(\beta t^2 + \frac{\alpha}{3}t^3 - \frac{1}{12}t^4\right) + O(\epsilon^2)$$

Show that this solution reproduces your answer from part (a) if $\beta = 8$ and $\alpha = 9\epsilon$ are plugged in and everything is re-sorted by powers of ϵ^n.

(d) The above solution is good for initial velocities being $O(1)$ or smaller. It does not work for singular initial velocities. Solve the problem with initial conditions $x(0) = 5/2$ and $x'(0) = 1/\epsilon$ using a singular scaling[2] for the solution, $x(t) = X(t)/\epsilon$. Find $X(t) \sim X_0(t) + \epsilon X_1(t) + \epsilon^2 X_2(t)$.

2. Consider the limit of $\epsilon \to 0$ for the equation

$$(x-3)^3 = 24\epsilon x^2 - 9\epsilon^2 x.$$

Solve for the single real-valued solution the by the iteration method to determine the first three terms in the asymptotic expansion: $x \sim \delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2$.
Hint: To find δ_0 you don't need to expand the LHS.

(continued)

[1] See the Maple worksheets
[2] $x = \delta X$ with $\delta = 1/\epsilon$. Convince yourself that this scaling yields a non-singular problem for $X(t)$.

3. (2010) Consider the algebraic equation for x with $\epsilon \to 0$:

$$\epsilon^9 x^3 - 4\epsilon^4 x^2 - 28\epsilon x - 56 - 8\epsilon^3 = 0.$$

Does this problem have a regular solution? Determine the leading order nontrivial term in the expansion of each of the three solutions. Show your checks for all six possibilities[3] for the leading order dominant balance.

4. (2011) Consider the algebraic equation for x with $\epsilon \to 0$:

$$\epsilon^4 x^3 - 4\epsilon x^2 + 8\epsilon^2 x + 36 = 0.$$

   (a) Let $x = \delta(\epsilon)X$ and use the method of dominant balance to find the two distinguished limits that yield the three solutions of this equation. Hint: let $\delta = \epsilon^\alpha$.

   (b) Write the rescaled equation for $X(\epsilon)$ for each of the distinguished limits and find the first two non-zero terms in the expansion of each solution, $X \sim X_0 + \delta_1 X_1$.

5. Consider the problem for $x(t)$ on $t \geq 0$ with $\epsilon \to 0$:

$$\frac{dx}{dt} - 5\epsilon x^2 = 3e^{2\epsilon t} \qquad x(0) = 4\cos(\epsilon).$$

   (a) Find the first three terms in the expansion of the solution, $x(t) \sim x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t)$. (Computer-assisted algebra optional)

   (b) Determine the range of times, $0 \leq t < O(\epsilon^\alpha)$, for which the terms in the expansion satisfy asymptotic ordering, i.e. $x_0 \gg \epsilon x_1 \gg \epsilon^2 x_2 \gg \cdots$.

   (We will discuss how to deal with related problems with competing limits ($\epsilon \to 0$ vs. $t \to \infty$) and the break-down of asymptotic ordering a bit later. But for this problem, we see that the solution from (a) is valid only for a limit range of times.)

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6. (Optional, extra credit) While most typical perturbation problems yield gauge functions as powers of epsilon, $\delta_n = \epsilon^n$ or $\delta = \epsilon^\alpha$, these are not the only things that can occur. This example shows one of the different types of $\delta(\epsilon)$'s that can also come up.

   Use the iteration method with $x \sim \delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1$ to find first two terms in the expansion of the *large positive* real solution of the equation

$$2x^2 e^{-5x} = 8\epsilon \qquad \epsilon \to 0.$$

   Note that this equation has three real roots; the two small roots are $x \sim \pm 2\epsilon^{1/2} + 10\epsilon$ (you don't need to do anything with these). The root we are interested in has $\delta_0(\epsilon) \to \infty$ (hence x is *large*) as $\epsilon \to 0$.

   Hint: Before starting the iteration approach, first take the logarithm, $\ln(\cdots)$, of both sides of the equation and use the properties of the log to separate terms as much as possible, $\ln(ab) = \ln(a) + \ln(b)$.

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[3] $6 = \binom{4}{2}$ combinations. Hint: Why don't you have to check $10 = \binom{5}{2}$ combinations? (Don't do 10, the other 4 can't work!)