Regular and Singular Perturbation Problems

1. **Computer-Assisted Algebra Policy** You may always use Maple or Mathematica (or other computer software) to help check your work on problem sets. For problems marked with “Computer-Algebra Recommended” (CAR), parts of the algebra may be somewhat long and tedious to work out by hand, so you can use computer-based assistance to carry out the work. For these problems, you must include copies of the PDF pages of the complete steps and computer-generated solutions. For other problems (Non-CAR), you should write out your solution process by hand.

0. Reading from Logan: Sections 3.1, 3.2. Also see the L05-demo.mws, L05-expansion.mws and L05-iteration.mws (expansion and iteration methods for the projectile motion example) files posted on Sakai for a guide on how to do the algebra for perturbation expansions via Maple.

1. **(Computer-Algebra Recommended)** Consider the projectile problem for \( \epsilon \to 0 \):

\[
\frac{d^2 x}{dt^2} = -\frac{1}{(1 + \epsilon x)^2}, \quad x(0) = 8, \quad x'(0) = \frac{\epsilon}{3}.
\]

(a) Obtain the first three terms in the expansion of the solution, \( x \sim x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) \).

Note that the initial height is different and the initial speed is slow.

(b) Let \( t^* \) be the time when the projectile lands on the ground, \( x(t^*) = 0 \). Use (a) to determine \( t^* = t_0 + \epsilon t_1 + \epsilon^2 t_2 + o(\epsilon^2) \).

Note: if you don’t keep enough terms in the AE’s for \( x \) and \( t^* \) your answers for \( t_0, t_1, t_2 \) may differ.

(c) The solution from Lecture 5\(^1\) can be used with IC’s \( x(0) = \beta = O(1) \) and \( x'(0) = \alpha = O(1) \) to yield

\[
x^{\alpha\beta}(t) = \left( \beta + \alpha t - \frac{1}{2} t^2 \right) + \epsilon \left( \beta t^2 + \frac{\alpha}{3} t^3 - \frac{1}{12} t^4 \right) + O(\epsilon^2)
\]

Show that this solution reproduces your answer from part (a) if \( \beta = 8 \) and \( \alpha = \epsilon/3 \) are plugged in and everything is re-sorted by powers of \( \epsilon^n \).

(d) The above solution is good for initial velocities being \( O(1) \) or smaller. It does not work for singular initial velocities. Solve the problem with initial conditions \( x(0) = 1 \) and \( x'(0) = 5/\epsilon \) using a singular scaling\(^2\) for the solution, \( x(t) = X(t)/\epsilon \). Find \( X(t) \sim X_0(t) + \epsilon X_1(t) + \epsilon^2 X_2(t) \).

2. Consider the limit of \( \epsilon \to 0 \) for the equation

\[
(x - 3)^3 = 24\epsilon x^2 - 9\epsilon^2 x.
\]

Solve for the single real solution the by the iteration method to determine the first three terms in the asymptotic expansion: \( x \sim \delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2 \).

Hint: To find \( \delta_0 \) you dont need to expand the LHS.

3. Consider the algebraic equation for \( x \) with \( \epsilon \to 0 \):

\[
\epsilon^6 x^3 - 5\epsilon^3 x^2 - 20\epsilon x + 60 = 0.
\]

Determine the leading order nontrivial term in the expansion of each of the three solutions. Show your checks for all six possibilities for the leading order dominant balance.

\(^1\)See the Maple worksheets

\(^2\)Convince yourself that this scaling yields a non-singular problem for \( X(t) \).
4. Consider the algebraic equation for \( x \) with \( \epsilon \to 0 \):

\[
\epsilon^4 x^3 - 4\epsilon x^2 + 8\epsilon^2 x + 36 = 0.
\]

(a) Let \( x = \delta(\epsilon)X \) and use the method of dominant balance to find the two distinguished limits that yield the three solutions of this equation. Hint: let \( \delta = \epsilon^\alpha \).

(b) Write the rescaled equation for \( X(\epsilon) \) for each of the distinguished limits and find the first two non-zero terms in the expansion of each solution, \( X \sim X_0 + \delta_1 X_1 \).

5. Consider the problem for \( v(t) \) on \( t \geq 0 \) with \( \epsilon \to 0 \):

\[
\frac{dv}{dt} + \epsilon v^2 + e^{\epsilon t} = 0 \quad v(0) = \cos(\epsilon).
\]

(a) Find the first three terms in the expansion of the solution, \( v(t) \sim v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t) \).

(b) Determine the range of times, \( 0 \leq t < O(\epsilon^\alpha) \), for which the terms in the expansion satisfy asymptotic ordering, i.e. \( v_0 \gg \epsilon v_1 \gg \epsilon^2 v_2 \gg \cdots \).

(We will discuss how to deal with related problems with competing limits (\( \epsilon \to 0 \) vs. \( t \to \infty \)) and the break-down of asymptotic ordering a bit later. But for this problem, we see that the solution from (a) is valid only for a limit range of times.)

6. (Optional, extra credit) While most typical perturbation problems yield gauge functions as powers of epsilon, \( \delta_0 = \epsilon^n \) or \( \delta = \epsilon^\alpha \), these are not the only things that can occur. This example shows one of the different types of \( \delta(\epsilon) \)'s that can also come up.

Use the iteration method with \( x \sim \delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 \) to find first two terms in the expansion of the large positive real solution of the equation

\[
2x^2e^{-5x} = 8\epsilon \quad \epsilon \to 0.
\]

Note that this equation has three real roots; the two small roots are \( x \sim \pm 2e^{1/2} + 10\epsilon \to 0 \) (you don’t need to do anything with these). The root we are interested in has \( \delta_0(\epsilon) \to \infty \) (hence \( x \) is large) as \( \epsilon \to 0 \).

Hint: Before starting the iteration approach, first take the logarithm, \( \ln(\cdots) \), of both sides of the equation and use the properties of the log to separate terms as much as possible, \( \ln(ab) = \ln(a) + \ln(b) \).