Math 577: Mathematical Modeling   Spring 2024

Problem Set 2 **Assigned Fri Jan 26** **Due Sat Feb 3**

Nondimensionalization and Scaling

0. Reading: Logan section 1.2. The notes for Lecture 4 may also be particularly useful.

1. Logan, Section 1.2.3 Exercises, page 43: Problem 7 ("A rocket blasts off...").

2. (2009) Consider the dimensional equation for $U(X, T)$:

$$\frac{\partial U}{\partial T} + (A + BU)\frac{\partial^2 U}{\partial X^2} + C\frac{\partial^3 U}{\partial X^3} = 0.$$

   U is a density (kg/m^3), X is position (m), and T is time (s).

   (a) (6 pts) Determine the units of the dimensional constants A, B, C.

   (b) (18 pts) Determine scalings $\underline{U}, \underline{L}, \underline{T}$ that yield a dimensionless equation for $u(x, t)$ with all coefficients normalized to 1.

3. (2022) Consider the third-order ODE for $X(T)$, given in dimensional form,

$$A\frac{d^3 X}{dT^3} + B\left(\frac{dX}{dT}\right)^2 + C\,T\,X^3 = D\,T^4$$

   where A, B, C, D are given positive dimensional constants.
   Let $X = \underline{L}\,x(t)$, with $T = \underline{T}\,t$ where $\underline{L}, \underline{T}$ are characteristic scaling constants.
   Select the characteristic scales to normalize as many terms as possible in the equation.

   For each of the <u>four</u> ways to scale the equations, determine $\underline{L}, \underline{T}$ and the single remaining <u>dimensionless parameter</u> (Π) in terms of the given quantities. Simplify your answers as much as possible.

4. (2020) Consider the dimensional equations for solutions $X(T), Y(T)$,

$$\frac{dX}{dT} = AXY - BY^3 \qquad \frac{dY}{dT} = -CX^2 + DT$$

   where A, B, C, D are given positive dimensional constants.
   Let $X(T) = \underline{X}\,x(t)$, $Y(T) = \underline{Y}\,y(t)$, and $T = \underline{T}\,t$ with $\underline{X}, \underline{Y}, \underline{T}$ being characteristic concentration and time-scales.
   Select the characteristic scales to normalize <u>**all**</u> of the coefficients in <u>**the x-rate equation**</u> and as many as possible in the other equation.

   For each of the <u>three</u> ways to scale the equations:

   (a) Write $\underline{X}, \underline{Y}, \underline{T}$ and the single remaining dimensionless parameter in terms of the given quantities. Simplify your answer as much as possible.

   (b) Write the scaled y-ODE with the dimensionless parameter.

(continued)

5. Consider the dimensional problem for projectile motion on the surface of the Earth:

$$\frac{d^2X}{dT^2} = -\frac{GM}{(R+X)^2} \qquad X(0) = 3\,\mathrm{m} \qquad X'(0) = -5\,\mathrm{m/sec}$$

Let $X(T) = \underline{L}\,x(t)$ and $T = \underline{T}\,t$. Consider the "small Earth" limit with

$$R \to 0 \qquad M = O(1).$$

(a) Choose your characteristic scales for $\underline{L}, \underline{T}$ to normalize as many terms as possible and ensure that none of the scaled coefficients in the problem blow-up in the limit.

(b) Write the scaled (normalized) problem, identify all remaining dimensionless parameters.

(c) Identify the limiting small dimensionless parameter ($\epsilon \to 0$) and write the leading order problem (where the limiting parameter is set to zero).

Hint: For (a) there are three possible choices for the scaled problem, you can use any of them.

6. Consider the dimensional equations describing a system of chemical reactions for the concentrations of three chemicals (X, Y, Z):

$$\begin{aligned} \frac{dX}{dT} &= A - BY & X(0) &= X_0 \\ \frac{dY}{dT} &= CX - DZ & Y(0) &= Y_0 \\ \frac{dZ}{dT} &= EY - FY^2 + GY^3 - HZ & Z(0) &= Z_0 \end{aligned} \qquad (1)$$

where A, B, C, D, E, F, G, H and the initial conditions X_0, Y_0, Z_0 are given dimensional constants. By scaling X, Y, Z, T (i.e. $X(T) = \underline{X}\,x(t)$ and similarly for the others) show that these equations can be non-dimensionalized to the form:

$$\begin{aligned} \frac{dx}{dt} &= \alpha - y & x(0) &= \mu \\ \beta\frac{dy}{dt} &= x - z & y(0) &= \sigma \\ \gamma\frac{dz}{dt} &= y - y^2 + \tfrac{1}{3}\delta y^3 - z & z(0) &= \omega \end{aligned} \qquad (2)$$

(a) Determine the characteristic scalings $\underline{X}, \underline{Y}, \underline{Z}, \underline{T}$ and the dimensionless parameters (Greek letters) in terms of A–H and X_0, Y_0, Z_0.

(b) If $\gamma = 0$ in (2), show that the problem reduces to a system of two ODE's for x and y: $dx/dt = p(x, y)$ and $dy/dt = q(x, y)$. Find the condition that the initial concentrations μ, σ, ω must satisfy in order to avoid a contradiction for this reduced problem.

(c) If $\beta = 0$ in (2), show that the problem can be reduced to a single ODE for y: $dy/dt = r(y)$. Find the two conditions that the initial concentrations μ, σ, ω must satisfy in order to avoid a contradiction for this reduced problem.

Hint: For parts (b,c), start each from the system in form (2), you will not need anything from part (a).