Math 577: Mathematical Modeling Spring 2024

Problem Set 1 **Assigned Fri Jan 19** **Due Sat Jan 27**

Dimensional analysis, scaling and self-similar solutions

0. **Reading**: Logan, Chapter 1: Section 1.1.1–1.1.3 and note that similarity solutions of PDEs are described on pages 19-21 (Example 1.9).

1. Logan, Section 1.1.4 Exercises, page 29: Problem 13. ("Did you ever wonder how fast....")

2. The fluid dynamics problem of how water drips from a faucet depends on several quantities: the acceleration due to gravity g [m/s^2], the density of water ρ [kg/m^3], the viscosity of water μ [kg/(m·s)], the surface tension of water σ [kg/s^2], the radius of the faucet nozzle R [m], and the speed of the water coming out of the faucet U [m/s].

   (a) Use the Buckingham Π theorem to determine the number of dimensionless parameters and write

   $$\Pi = g^A \rho^B \mu^C \sigma^D R^E U^F$$

   to determine the set of equations relating the exponents A-F.

   (b) Solve for A, B, C in terms of D, E, F. (Hint: Use Gaussian elimination on the eqns from (a))

   (c) The choice of dimensionless parameters to describe a problem is not unique. This problem can be described in terms of the set of "historically named" parameters (from engineering/fluid dynamics):

   $$\mathrm{We} = \frac{\rho U^2 R}{\sigma} \qquad \mathrm{Oh} = \frac{\mu}{\sqrt{\rho R \sigma}} \qquad \mathrm{Ga} = \frac{\rho^2 g R^3}{\mu^2} \qquad \{\text{Weber, Ohnesorge, and Galileo numbers}\}$$

   or in terms of the set of

   $$\mathrm{Re} = \frac{\rho U R}{\mu} \qquad \mathrm{Bo} = \frac{\rho g R^2}{\sigma} \qquad \mathrm{Ca} = \frac{\mu U}{\sigma} \qquad \{\text{Reynolds, Bond, and Capillary numbers}\}$$

   Show that the $A - F$ values for each set of 3 parameters satisfy the equations from (b).

   (d) What test from linear algebra can you use to show that the sets of parameters in (c) are independent? (Do it...)

3. A fictional industrial process operates at the following setting for system parameters: A [kg·mol/(m·s)], B [m·s^2/(kg·mol)], C [kg^3/m^3] and D [mol·s^3] (m=meters, s=seconds, kg=kilograms, mol=moles).

   (a) Count the number of given quantities and dimensional units. How many dimensionless parameters do you expect?

   (b) Carry out the linear algebra calculation to determine the form of the dimensionless parameter(s).

   (c) Due to fictional cutbacks, D has been reduced from $D = 10$ to $D = 1$, but the system should be kept operating in the same state. Explain how to do this by changing one other parameter at a time (i.e. change A but leave the same BC, or change B with same AC, change C with same AB).

4. (2022) Consider the PDE for $u(x,t)$ on $0 \le x < \infty$ with $t > 0$:

   $$t\frac{\partial u}{\partial t} = 4x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + 15t^3 \qquad \text{with the boundary condition at } x=0\text{:} \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 7t$$

   (a) Determine α, β in the form of the similarity solution, $u(x,t) = t^{-\beta} f(\eta)$ with $\eta = xt^{\alpha}$.

   (b) Write the ODE and BC for $f(\eta)$. Simplify as much as possible.

   (c) Determine the solution $f(\eta)$ and then $u(x,t)$. (Hint: Inhomogeneous Cauchy Euler ODE)

(continued)

5. (2021) Consider the PDE's for $u(x,t), w(x,t)$ on $0 \le x < \infty$ with $t \ge 1$:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = w^3\,, \qquad \frac{\partial^2 w}{\partial t^2} = u^2\,\frac{\partial^2 w}{\partial x^2} + \frac{x}{w^5}\,,$$

with boundary conditions at $x = 0$:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 5t^{\sigma}\,, \qquad \left.\frac{\partial w}{\partial t}\right|_{x=0} = 12t^{1/3}$$

(a) Determine the value of the constant σ so that the problem is scale invariant.

(b) Determine the scalings U, W, L for $u = U\tilde{u},\ w = W\tilde{w},\ x = L\tilde{x}$ in terms of the timescale T in $t = T\tilde{t}$

(c) Identify α, β, γ in the similarity solution, $u(x,t) = t^{-\beta} f(\eta)$, $w(x,t) = t^{-\gamma} g(\eta)$ with $\eta = xt^{\alpha}$.

(d) Write the ODE's and BC's for $f(\eta), g(\eta)$. Simplify as much as possible.

6. The following problem describes the height of an axisymmetric drop of viscous fluid ($z = h(r,t) \ge 0$) spreading on a dry flat surface ($z = 0$) due to the influence of gravity:

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right) \qquad \text{on } 0 \le r < \infty$$

subject to the conditions:

$$\left.\frac{\partial h}{\partial r}\right|_{r=0} = 0 \quad \text{(symmetry)}, \qquad 2\pi \int_0^{\infty} h(r,t) r\, dr = 4\pi \cdot 3^{4/3} \quad \text{(volume)}.$$

(a) This problem is scale invariant; find α, β for its similarity solution, $h(r,t) = t^{-\beta} f(\eta)$ where $\eta = rt^{\alpha}$.

(b) The ODE for $f(\eta)$ can be written as $\quad -\beta f + \alpha\eta\dfrac{df}{d\eta} = \dfrac{1}{\eta}\dfrac{d}{d\eta}\left(\eta f^3 \dfrac{df}{d\eta}\right).$

If your values of α, β are correct, you can integrate both sides of the equation after multiplying across by η. Use the symmetry boundary condition to set the value of the first constant of integration. Then you can integrate again to get the solution in the form $f^3(\eta) = A - B\eta^2$; find B.

(c) The integral condition above sets the volume of the droplet. The solution of this problem has a positive height on a finite range $0 \le r \le R(t)$ with $h(R(t),t) = 0$ defining a moving "edge" of the drop (also called the "contact line", "interface" or "moving boundary"). The surface is dry beyond the contact line (no fluid), so the solution is truncated to zero, $h \equiv 0$ for $r > R(t)$.

Determine the similarity solution $h(r,t)$ and the moving boundary $R(t)$ satisfying the integral condition on the volume.

Hint: Note that $\int_0^\infty hr\,dr = \int_0^R hr\,dr$ since $h = 0$ for $r > R$, and find the value of the similarity variable, $\eta = S$, that corresponds to the interface $R(t)$.