0. **Reading**: Logan, Chapter 1: Section 1.1.1–1.1.3 and note that similarity solutions of PDEs are described on pages 19-21 (Example 1.9).


2. Consider the system of two dimensional equations for unknowns \( x, y \):

\[
xy + Ax^2 + By^3 = 0 \quad y + \frac{C}{x^2} + Dxy^2 = 0
\]

where \( A, B, C, D \) are given positive dimensional constants. The dimensions of \( y \) are \([y] = L^2T^{-1}\) and the dimensions of \( A \) are \([A] = L^{-1}T^{-3}\) (where \( L \) is length and \( T \) is time).

(a) Determine the dimensions for \( x, B, C, D \).

(b) Scale \( x = X \tilde{x} \) and \( y = Y \tilde{y} \). There are several (6?) ways to select characteristic scales for \( X, Y \) but just calculate two of these cases: determine the \( X, Y \) scales and the dimensionless \( \Pi \) coefficient parameters in terms of \( ABCD \) for (i) when all terms in the first eqn are normalized, and (ii) when all terms in the second eqn are normalized.

3. The fluid dynamics problem of how water drips from a faucet depends on several quantities: the acceleration due to gravity \( g \) [m/s\(^2\)], the density of water \( \rho \) [kg/m\(^3\)], the viscosity of water \( \mu \) [kg/(m·s)], the surface tension of water \( \sigma \) [kg/s\(^2\)], the radius of the faucet nozzle \( R \) [m], and the speed of the water coming out of the faucet \( U \) [m/s].

(a) Use the Buckingham \( \Pi \) theorem to determine the number of dimensionless parameters and write

\[
\Pi = g^A \rho^B \mu^C \sigma^D R^E U^F \]

to determine the set of equations relating the exponents \( A-F \).

(b) Solve for \( A, B, C \) in terms of \( D, E, F \). (Hint: Use Gaussian elimination on the eqns from (a))

(c) The choice of dimensionless parameters to describe a problem is not unique. This problem can be described in terms of the set of “historically named” parameters (from engineering/fluid dynamics):

\[
\begin{align*}
\text{We} & = \frac{\rho U^2 R}{\sigma} & \text{Oh} & = \frac{\mu}{\sqrt{\rho R \sigma}} & \text{Ga} & = \frac{\rho^2 g R^3}{\mu^2} & \text{Weber, Ohnesorge, and Galileo numbers} \\
\text{Re} & = \frac{\rho U R}{\mu} & \text{Bo} & = \frac{\rho g R^2}{\sigma} & \text{Ca} & = \frac{\mu U}{\sigma} & \text{Reynolds, Bond, and Capillary numbers}
\end{align*}
\]

or in terms of the set of

Show that the \( A-F \) values for each set of 3 parameters satisfy the equations from (b). How can you show that the parameters are independent?

4. Consider the partial differential equation for \( u(x, t) \):

\[
9 \frac{\partial^2 u}{\partial t^2} + u \frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x^3}.
\]

(a) Determine the scalings \( U, L \) for \( u, x \) in terms of the timescale \( T \) to show that the PDE has a scale invariant solution.

(b) Identify \( \alpha, \beta \) in the similarity solution, \( u(x, t) = t^\alpha f(\eta) \) with \( \eta = xt^\beta \).

(c) Consider \( u(x, t) \) on \( x \geq 0 \). Determine \( \sigma \) to make the boundary condition scale invariant: \( \frac{\partial^2 u}{\partial x^2} \bigg|_{x=0} = 4t^\sigma \).
5. Consider the partial differential equations for $u(x,t), v(x,t)$:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} (3uv) + 4xu, \quad \frac{\partial v}{\partial t} = \frac{\partial^2}{\partial x^2} (5u^3) + 6x^\sigma t.
\]

(a) Determine the value of the constant $\sigma$ so that the problem is scale invariant.

(b) Determine the scalings $U, V, L$ for $u = U\tilde{u}, v = V\tilde{v}, x = L\tilde{x}$ in terms of the timescale $T$ in $t = T\tilde{t}$

(c) Identify $\alpha, \beta, \gamma$ in the similarity solution, $u(x,t) = t^\alpha f(\eta), v(x,t) = t^\gamma g(\eta)$ with $\eta = xt^\beta$.

(d) Write the ODE’s for $f(\eta), g(\eta)$.

6. The following problem describes the height of an axisymmetric drop of viscous fluid ($h = h(r,t) \geq 0$) spreading on a dry surface ($h = 0$) due to the influence of gravity:

\[
\frac{\partial h}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( rh^3 \frac{\partial h}{\partial r} \right) \quad \text{on} \quad 0 \leq r < \infty
\]

subject to the boundary and integral conditions:

\[
\frac{\partial h}{\partial r} \bigg|_{r=0} = 0 \quad \int_0^\infty h(r,t) r \, dr = 2 \cdot 3^{4/3}.
\]

(a) This problem is scale invariant; find $\alpha, \beta$ for its similarity solution, $h(r,t) = t^\alpha f(\eta)$ where $\eta = rt^\beta$.

(b) The ODE for $f(\eta)$ can be written as

\[
\alpha f + \beta \eta \frac{df}{d\eta} = \frac{1}{\eta} \frac{d}{d\eta} \left( \eta f^3 \frac{df}{d\eta} \right).
\]

If your values of $\alpha, \beta$ are correct, you can integrate both sides of the equation (after multiplying across by $\eta$ [the right “integrating factor” for this equation]). Use the boundary condition to set the value of the first constant of integration. Then you can integrate again to get the solution in the form $f^3(\eta) = A - B\eta^2$; find $B$.

(c) The integral condition above sets the volume of the droplet. The solution of this problem is positive on a finite range $0 \leq r \leq r_*(t)$ with $h(r_*(t),t) = 0$ defining a moving “edge” of the drop (also called the “contact line”, “interface” or “moving boundary”). The surface is dry beyond the contact line (no fluid), so the solution is truncated to zero, $h \equiv 0$ for $r > r_*(t)$.

Determine the similarity solution $h(r,t)$ and the moving boundary $r_*(t)$ satisfying the integral condition on the volume.

Hint: Note that $\int_0^\infty hr \, dr = \int_0^{r_*} hr \, dr$ since $h = 0$ for $r > r_*$, and let $\eta_*$ correspond to the interface $r_*(t)$. 