Math 575: Mathematical Fluid Dynamics — Spring 2022

Homework 7 — **Assigned: Fri Feb 25** — **Due: Fri Mar 18**

Complex potential flows (concluded)

0. Reading: Acheson, sections 4.6-4.13.

1. The moment (torque, with respect to the origin) on a solid object with boundary curve C in a steady 2D potential flow is given by the line integral $M = \oint_C xp\,dx + yp\,dy$. Show that it can be expressed in terms of the complex integral,

$$M = -\frac{\rho}{2}\mathrm{Re}\left(\oint_C z\left(\frac{dw}{dz}\right)^2 dz\,.\right)$$

(See Acheson, page 153, problem 4.5)

2. Consider an inviscid irrotational incompressible fluid flow occupying the region $x^2+y^2 \geq a^2$ outside a rigid circular cylinder of radius a with a source of strength Q at $(b,0)$, where $b > a$, and a circulatory flow around the cylinder as if due to a vortex of strength Γ at $(0,0)$.

Explain why the complex potential is

$$w(z) = \frac{Q}{2\pi}\log(z-b) + \frac{Q}{2\pi}\log\left(\frac{a^2}{z} - b\right) - \frac{i\Gamma}{2\pi}\log z$$

Calculate dw/dz, and use Blasius' theorem to find the force (F_x, F_y) on the cylinder.

3. Recall that the complex potential $w(z) = \sqrt{z}$ (with a branch cut along the positive x-axis) describes the steady 2-D potential flow with parabolic streamlines around an semi-infinite flat plate (the positive x-axis). Use the Blasius integral to calculate the force imposed on the plate by the flow.
Hint: Your path of integration will be a loop around the plate. Consider the contour $C = C_1+C_2+C_3$ with $\epsilon \geq 0$,

$$\begin{array}{llll}
C_1 = (\infty,\epsilon) \to (0,\epsilon) & z = x + i\epsilon & dz = dx & x : \infty \to 0, \\
C_2 = (0,\epsilon) \to (0,-\epsilon) & z = \epsilon e^{i\theta} & dz = i\epsilon e^{i\theta}\,d\theta & \theta : \pi/2 \to 3\pi/2, \\
C_3 = (0,-\epsilon) \to (\infty,-\epsilon) & z = x - i\epsilon & dz = dx & x : 0 \to \infty.
\end{array}$$

4. Recall from lecture that uniform flow with speed U and angle of inclination α around a circular cylinder of radius a with additional circulation Γ is described by the potential

$$w(s) = U\left(se^{-i\alpha} + \frac{a^2}{se^{-i\alpha}}\right) - \frac{i\Gamma}{2\pi}\log s$$

and using the Joukowski transformation

$$z = s + \frac{a^2}{s}$$

this can be transformed to yield the complex velocity, dw/dz, for the flow around a flat plate, $-2a \leq z \leq 2a$ (portion of the x-axis, corresponding to $s = ae^{i\theta}$).

(a) Determine the stagnation points of the flow in terms of s from $w(s)$. Determine the conditions on Γ under which the stagnation points lie on the plate. Determine formulas for the z values of the stagnation points. Identify which lies on the upper/lower surfaces of the plate (see left figure below).

(b) Show that the speed of the flow, $|dw/dz|$, diverges at the tips of the plate, $z = \pm 2a$.

There is a value of the circulation Γ which eliminates the flow singularity for $s = a$ (see right figure below); this is called the **<u>Kutta condition</u>**. Determine the formula for this Γ. Use the Kutta-Joukowski Lift theorem to determine the force on the plate; write it in terms of $\mathbf{F} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}$.

Using the Kutta condition, substitute $s = a + \epsilon$ into your expression for dw/dz to determine the first term in the Taylor series expansion for the complex velocity at $z = 2a$, $dw/dz = f_0 + O(\epsilon)$ as $\epsilon \to 0$.