Complex potential flows (concluded)

0. Reading: Acheson, sections 4.6-4.13.

1. The moment (torque, with respect to the origin) on a solid object with boundary curve $C$ in a steady 2D potential flow is given by the line integral $M = \int_C xp \, dx + yp \, dy$. Show that it can be expressed in terms of the complex integral,

$$M = -\frac{\rho}{2} \text{Re} \left( \oint_C z \left( \frac{dw}{dz} \right)^2 \, dz \right).$$

(See Acheson, page 153, problem 4.5)

2. Consider an inviscid irrotational incompressible fluid flow occupying the region $x^2 + y^2 \geq a^2$ outside a rigid circular cylinder of radius $a$ with a source of strength $Q$ at $(b,0)$, where $b > a$, and a circulatory flow around the cylinder as if due to a vortex of strength $\Gamma$ at $(0,0)$.

Explain why the complex potential is

$$w(z) = \frac{Q}{2\pi} \log(z - b) + \frac{Q}{2\pi} \log \left( \frac{a^2}{z} - b \right) - \frac{i\Gamma}{2\pi} \log z.$$

Calculate $dw/dz$, and use Blasius’ theorem to find the force $(F_x, F_y)$ on the cylinder.

3. Recall that the complex potential $w(z) = \sqrt{z}$ (with a branch cut along the positive $x$-axis) describes the steady 2-D potential flow with parabolic streamlines around an semi-infinite flat plate (the positive $x$-axis). Use the Blasius integral to calculate the force imposed on the plate by the flow.

Hint: Your path of integration will be a loop around the plate. Consider the contour $C = C_1 + C_2 + C_3$ with $\epsilon \geq 0$,

$$C_1 = (\infty, \epsilon) \to (0, \epsilon) \quad z = x + i\epsilon \quad dz = dx \quad x : \infty \to 0,$$
$$C_2 = (0, \epsilon) \to (0, -\epsilon) \quad z = \epsilon e^{i\theta} \quad dz = i\epsilon e^{i\theta} \, d\theta \quad \theta : \pi/2 \to 3\pi/2,$$
$$C_3 = (0, -\epsilon) \to (\infty, -\epsilon) \quad z = x - i\epsilon \quad dz = dx \quad x : 0 \to \infty.$$

4. Recall from lecture that uniform flow with speed $U$ and angle of inclination $\alpha$ around a circular cylinder of radius $a$ with additional circulation $\Gamma$ is described by the potential

$$w(s) = U \left( se^{-i\alpha} + \frac{a^2}{se^{-i\alpha}} \right) - \frac{i\Gamma}{2\pi} \log s$$

and using the Joukowski transformation

$$z = s + \frac{a^2}{s}$$

this can be transformed to yield the complex velocity, $dw/dz$, for the flow around a flat plate, $-2a \leq z \leq 2a$ (portion of the $x$-axis, corresponding to $s = ae^{i\theta}$).

(a) Determine the stagnation points of the flow in terms of $s$ from $w(s)$. Determine the conditions on $\Gamma$ under which the stagnation points lie on the plate. Determine formulas for the $z$ values of the stagnation points. Identify which lies on the upper/lower surfaces of the plate (see left figure below).
(b) Show that the speed of the flow, $|dw/dz|$, diverges at the tips of the plate, $z = \pm 2a$.

There is a value of the circulation $\Gamma$ which eliminates the flow singularity for $s = a$ (see right figure below); this is called the **Kutta condition**. Determine the formula for this $\Gamma$. Use the Kutta-Joukowski Lift theorem to determine the force on the plate; write it in terms of $F = F_x \mathbf{i} + F_y \mathbf{j}$.

Using the Kutta condition, substitute $s = a + \epsilon$ into your expression for $dw/dz$ to determine the first term in the Taylor series expansion for the complex velocity at $z = 2a$, $dw/dz = f_0 + O(\epsilon)$ as $\epsilon \to 0$. 

\[ \text{Diagram: Flow field with Kutta condition applied.} \]