Complex variables methods for 2D potential flows (part 1 of 2)

0. Reading: Acheson, sections 4.2, 4.3, 4.4, 4.6. For more about Schwarz-Christoffel mappings (especially computing for more complicated problems that can’t be solved explicitly), see the book by T. A. Driscoll and L. N. Trefethen.

1. A Schwarz-Christoffel mapping function was constructed as

\[ z = h(s) = \int_0^s \frac{(t-2)(t-4)}{t(t-5)} \, dt \]

where \( s_1 = 0 \) maps to \( z_1 = 0 \), and \( s_2 = 2, s_3 = 4, s_4 = 5 \) map to some \( z_k \) for \( k = 2, 3, 4 \).

(a) Based on the form of the factors in the mapping function, draw a sketch of the boundary of the domain in the \( z \)-plane that is the image of the upper-half plane from the complex \( s \)-plane. Leave the values of the other \( z_k \) corners un-specified, but make sure the angles and orientations in your sketch are correct.

(b) Numerically compute the corners \( z_2, z_3, z_4 \). You can use Maple, Mathematica or any other means to compute these values.

(c) If your values from (b) are not consistent with the geometry of your sketch from (a), you must conclude that the right branches (\( \pm \sqrt{\cdot} \)) were not used in the computations. Recall the derivation of the Schwarz-Christoffel formula from class, and use the ideas there (use of absolute values, phase angles and breaking up the integral by segments along the real \( s \)-axis) to find the correct values for the \( z_k \)’s.

2. (a) As derived in class, the complex potential for a dipole singularity (sometimes called a ‘doublet’) is \( w(z) = m/z \) with \( m \) being a constant. Determine the streamfunction \( \psi(x, y) \). Show that the streamlines are circles through the origin with centers on the \( y \)-axis. Sketch the streamlines and equipotential curves.

(b) Consider the combined flow due to a dipole with \( m = 2 \) at the origin and a uniform flow with \( u = 0, v = 1 \). Find the stagnation points, determine the streamfunction \( \psi(x, y) \) and sketch the streamlines.

3. An incompressible irrotational inviscid fluid flow occupies the upper half of the \( xy \) plane, \( y \geq 0 \), with an impermeable wall at \( y = 0 \). There is a uniform flow, speed \( U \), in the positive \( x \)-direction, and a source of strength \( Q \) at \( (x, y) = (0, b) \). Write the complex potential \( w(z) \), and calculate \( dw/dz \). Let \( \alpha = Q/(2\pi bU) \). Show that if \( |\alpha| > 1 \) there are two stagnation points, both on the wall, while if \( |\alpha| < 1 \) there is only one in the fluid, a distance \( b \) from the origin \( (0,0) \).

4. (Dynamics of interacting flow singularities) Complex potentials \( w(z) \) can be written for flows with contributions from point singularities (sources, sinks, vortices, ...) and (non-singular) smooth background flows (from conformal mappings, etc). For some applications, rather than having these points be at specified fixed locations, the singularities can be treated as a type of particle that move in response to the flow.

Passive scalar (or “tracer”) particles, are carried by the flow and do not influence the flow. So, for a passive particle \( \vec{x}_1(t) = (x_1(t), y_1(t)) \), the equations for its pathline would be

\[ \frac{dx_1}{dt} = u(x_1, y_1) \quad \frac{dy_1}{dt} = v(x_1, y_1) \]

or combined into complex form, \( z_1(t) = x_1(t) + iy_1(t) \) and \( f(z) = u - iv \)

\[ \frac{dx_1}{dt} - i \frac{dy_1}{dt} = f(z_1) \quad \Rightarrow \quad \frac{d\bar{z}_1}{dt} = f(z_1) \]

\[ \text{Always “fact-check” numerical results!} \]
where \( f(z) = dw/dz \). Since the final form of this ODE contains both \( z \) and \( \bar{z} \), it is not analytic, so can’t be solved “analytically” using complex variable methods, so in practice this is used to write the ODE’s for \( dx/dt \) and \( dy/dt \) after splitting \( dw/dz \) into real and imaginary parts, \( u - iv \).

Sources, sinks, and vortices are simple examples of “active particles” (sometimes called “swimmers”) that do change the flow when they move because they are the centers of self-generated flow fields. It can be shown (see for example C. Marchioro and M. Pulvirenti, Mathematical Theory of Incompressible Nonviscous Fluids) that these points will move according to the velocity generated by all other contributions other than themselves, namely each particle does not feel the flow field that it generates (i.e. “no self-interaction”). So, if \( z_1(t) \) is a source or vortex with \( w_1(z) = c \log(z - z_1(t)) \) and the total flow is described by \( W(z) \) from the sum of all contributions (all sources, sinks, vortices, dipoles, images, uniform flow, other background flows...) then the motion of \( z_1(t) \) is given by

\[
\frac{d\bar{z}_1}{dt} = W'_1(z_1) \quad \text{where } W_1(z) = W(z) - w_1(z).
\]

Apply this background to:

(a) Consider a source of strength \( Q \) at a point \( z_1 \) in the upper half of the complex plane. Write down the complex potential for the flow in the upper half plane with a wall present at \( y = 0 \).

Let the source be free to move. Determine the equations of motion for \( x_1(t) \) and \( y_1(t) \) and solve them starting from initial condition \( z_1(t = 0) = a + ib \) with \( b > 0 \).

(b) Consider a vortex of strength \( \Gamma_1 \) at a point \( z_1 \) outside the solid circular cylinder \( |z| \leq a \). Write down the complex potential for this flow.

Let the vortex be free to move. Determine the equation of motion for \( \bar{z}_1(t) \). Show that the vortex will orbit the cylinder on a concentric circle. Determine the vortex’s angular velocity \( \Omega \) if the vortex starts from \( z_1(t = 0) = b > a \).

(c) Consider two interacting vortices,

\[
w_1(z) = -\frac{i\Gamma_1}{2\pi} \log(z - z_1(t)), \quad w_2(z) = -\frac{i\Gamma_2}{2\pi} \log(z - z_2(t)),
\]

where the position of each is \( z_k(t) = x_k(t) + iy_k(t) \). Each vortex moves according to the total velocity field at its position excluding its own self-influence. Write down the equations of motion for the vortices.

i. Show that the separation distance, \( L = |z_1 - z_2| \), remains constant.

ii. Show that if \( \Gamma_1 + \Gamma_2 \neq 0 \) then the center of mass of the vortices,

\[
Z = \frac{\Gamma_1 z_1 + \Gamma_2 z_2}{\Gamma_1 + \Gamma_2},
\]

remains constant.

iii. Show that the vortices move on concentric circles centered at \( Z \) and determine their common angular velocity \( \Omega \). What are their paths if \( \Gamma_1 + \Gamma_2 = 0 \)?

iv. Show that the equations of motion can be written in term of the Hamiltonian,

\[
H = -\frac{\Gamma_1\Gamma_2}{2\pi} \ln(L),
\]

as

\[
\frac{dx_k}{dt} = \frac{1}{\Gamma_k} \frac{\partial H}{\partial y_k}, \quad \frac{dy_k}{dt} = -\frac{1}{\Gamma_k} \frac{\partial H}{\partial x_k}, \quad \text{for } k = 1, 2.
\]