Math 575: Mathematical Fluid Dynamics — Spring 2022

Homework 5 **Assigned: Fri Feb 11** **Due: Fri Feb 18**

Inviscid, Incompressible, Irrotational flows

0. Reading: Acheson, sections 1.3, 1.5, 4.1, 4.2, 4.3.
1. Consider each two-dimensional steady flow:
   (a) $u(x,y) = 2x - 6x^2 + y$, $v(x,y) = x + 3y^2$. Determine the velocity potential. Show that a streamfunction does not exist.
   (b) $u(x,y) = x^2 - 9y^2 + 5$, $v(x,y) = -2xy - 2x$. Determine the streamfunction. Show that a velocity potential does not exist.
   (c) Determine $u(x,y)$ and the complex potential $w(z)$ for the incompressible, irrotational flow with $v(x,y) = 4x - 6y - 6xy - 8$.
   (d) Some of the previous parts asked about examples with failures to be incompressible or irrotational, but there wasn't one for not being inviscid. Consider smooth steady 2D flow $\vec{\mathbf{u}} = (u(x,y), v(x,y), 0)$:
      i. For incompressible flows, explain which form of the Euler momentum balance shows all $\vec{\mathbf{u}}$ are acceptable without the need for any calculations.
      ii. For steady compressible flows, find the conditions that ρ, p must satisfy so that any smooth $\vec{\mathbf{u}}$ can work.
2. Consider the real velocity potential $\phi(x,y) = (x^2+y^2)^{1/4}\cos\left(\frac{1}{2}\tan^{-1}(y/x)\right)$.
   Show that the streamlines are a one-parameter family of parabolas.
   Hint: Show that the streamlines satisfy a nonlinear homogeneous-type first order ODE, $dy/dx = F(y/x)$ (see summary sheet). Use of computer-aided algebra programs is allowed.
3. Consider the 2D unsteady flow for a moving irrotational vortex, $\vec{\mathbf{u}}(x,y,t) = \left(\dfrac{-y}{(x-at)^2+y^2}, \dfrac{x-at}{(x-at)^2+y^2}\right)$
   (a) Determine the pressure $p(x,y,t)$.
   (b) Show that this flow satisfies the 2D unsteady Euler equations
   (use of computer algebra is encouraged, provide complete printouts).
4. Show that any (non-degenerate) stagnation point[1] of a steady two-dimensional incompressible irrotational flow, $\left\{\dfrac{dx}{dt} = u(x,y) \text{ and } \dfrac{dy}{dt} = v(x,y)\right\}$, must be a hyperbolic saddle point, with $\lambda_+ = -\lambda_-$ in its linearized stability (phase plane) analysis.
5. About streamfunctions $\psi(x,y,t)$ for two-dimensional incompressible flows:
   (a) Show that the velocity field automatically satisfies the incompressibility condition, for any ψ.
   (b) Determine the scalar vorticity in terms of the streamfunction.
   (c) Show that the 2D inviscid scalar vorticity equation can be written as $\dfrac{\partial\omega}{\partial t} + \dfrac{\partial\psi}{\partial y}\dfrac{\partial\omega}{\partial x} - \dfrac{\partial\psi}{\partial x}\dfrac{\partial\omega}{\partial y} = 0$
   (d) Show that all steady solutions from (c) satisfy $\nabla^2\psi = q(\psi)$ for some function q.
      i. What choice of q corresponds to irrotational flow?
      ii. Solve $\nabla^2\psi = -\psi$ for axisymmetric solutions, $\psi = \psi(r)$.
      iii. Solve $\nabla^2\psi = e^\psi$ in one-dimension, $\psi = \psi(x)$.
   Hints: $ab - cd = \det(\mathbf{A})$ and the determinant of a 2×2 matrix is zero if the rows are proportional to each other.
   (e) Extend your result from (c) to the viscous case to write the "vorticity-streamfunction" form of the 2-D Navier-Stokes equations.
   Describe how you might apply no-slip and no-flux boundary conditions if you were to use only ω, ψ.

[1] a non-degenerate stagnation point has the velocity vector being zero (as usual), but also has that the gradient of the velocity is not zero.