Bernoulli’s theorem and Vorticity

0. Reading: Acheson, sections 1.3, 1.4, 1.5 (and other course hand-outs will be useful).

1. Acheson, page 24, problem 1.5.

2. Vector identities:
   (a) Use index notation and the equality of mixed partial derivatives \( f_{ij} = f_{ji} \) for any smooth \( f(x) \) to show that the divergence of the curl is zero for all smooth vector fields, \( \nabla \cdot (\nabla \times \mathbf{u}) = 0 \).
   (b) Show that for any nice \( f, g \) that \( \nabla \cdot (\nabla f \times \nabla g) = 0 \). Hint: you can use your results from (a).
   (c) Use index notation to show the identity for any smooth vector field \( \mathbf{u}(\mathbf{x}) \),

   \[
   (\nabla \times \mathbf{u}) \times \mathbf{u} = \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{2} \nabla (|\mathbf{u}|^2).
   \]
   (d) Use index notation to derive (starting from LHS, obtain RHS) the vector identity \( \text{curl} (\text{curl} \mathbf{F}) = \text{grad} (\text{div} \mathbf{F}) - \nabla^2 \mathbf{F} \). Use this identity to show that for three-dimensional incompressible, irrotational flows, each component of the velocity is a harmonic function.

3. Beltrami flows: Let \( \mathbf{u}(\mathbf{x}) \) be a steady incompressible flow. The velocity field \( \mathbf{u} \) is called a Beltrami flow if the vorticity is everywhere parallel to the velocity. Note that parallel vectors are scalar multiples of each other, the scalar can be a function: \( \mathbf{\omega} = \phi(\mathbf{x})\mathbf{u} \).
   (a) Show that Beltrami flows are solutions of the momentum balance in the Euler equations. What is the pressure for the flow?
   (b) Determine the compatibility condition between \( \phi, \mathbf{u} \) needed to satisfy the incompressibility condition.

4. Helicity: Let \( \mathbf{u}(\mathbf{x}, t) \) be a solution of the Euler equations on \( \mathbb{R}^3 \), show that the helicity \( \mathcal{H} \) is a constant,

   \[
   \mathcal{H}(t) = \iint_{-\infty}^{\infty} \mathbf{u} \cdot \mathbf{\omega} \, dV.
   \]
   Assume that \( |\mathbf{u}| \to 0 \) as \( |\mathbf{x}| \to \infty \).

5. Bernoulli’s theorem:
   (a) In lecture, we derived Bernoulli’s theorem for solutions of Euler’s equations, which roughly says “\( H = \frac{p}{\rho} + \frac{1}{2} |\mathbf{u}|^2 + \frac{V}{\rho} \text{ is constant} \)” This understanding is a bit too “rough”; consider the two steady 2D flows (velocities and pressures):

   \[
   \mathbf{u}_1 = (-y, x) \quad p_1 = \frac{1}{2}(x^2 + y^2)
   \]

   \[
   \mathbf{u}_2 = \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \quad p_2 = 3 - \frac{1}{2(x^2 + y^2)}
   \]

   By direct substitution, show that both are solutions of the steady Euler equations with no body forces and density \( \rho \equiv 1 \). Show that they have distinctly different \( H \)’s. Explain the meaning of “is constant” for the versions of Bernoulli’s theorem appropriate to each flow to resolve the apparent conflict. Hint: characterize each flow (what adjectives apply?).
   (b) For each flow, calculate the circulation, \( \Gamma = \oint_C \mathbf{u} \cdot d\mathbf{x} \), on the unit circle, \( C : \{ |\mathbf{x}| = 1 \} \).
6. Polar coordinates: (use of computer-aided algebra is allowed/encouraged)

(a) (Cylindrical) Using the fact that the Laplacian of a scalar function in polar coordinates is
\[ \nabla^2 \phi(r, \theta, z) = r^{-1} \partial_r (r \partial_r \phi) + r^{-2} \partial_{\theta \theta} \phi + \partial_{zz} \phi, \]
derive the \( \hat{e}_r, \hat{e}_\theta, \hat{e}_z \) components of the Laplacian
of a vector field \( \mathbf{F} \) given in polar form as
\[ \mathbf{F} = F_r(r, \theta, z) \hat{e}_r + F_\theta(r, \theta, z) \hat{e}_\theta + F_z(r, \theta, z) \hat{e}_z. \]

(b) (Spherical) Consider spherical coordinates
\[ x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi. \]

i. Find expressions for the unit vectors \( \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi. \)

ii. Write the velocity as
\[ \mathbf{u} = u_\rho(\rho, \theta, \phi) \hat{e}_\rho + u_\theta(\rho, \theta, \phi) \hat{e}_\theta + u_\phi(\rho, \theta, \phi) \hat{e}_\phi, \]
express \( u_x(x, y, z, t), u_y(x, y, z, t), u_z(x, y, z, t) \) in terms of the spherical-coordinate velocity component functions. Also, find \( u_\rho, u_\theta, u_\phi \) in terms of the rectangular velocity components.