Math 575: Mathematical Fluid Dynamics — Spring 2022

Homework 4 — **Assigned: Fri Feb 4** — **Due: Fri Feb 11**

Bernoulli's theorem and Vorticity

0. Reading: Acheson, sections 1.3, 1.4, 1.5 (and other course hand-outs will be useful).

1. Acheson, page 24, problem 1.5.

2. Vector identities:

   (a) Use index notation and the equality of mixed partial derivatives ($f_{,ij} = f_{,ji}$ for any smooth $f(\mathbf{x})$) to show that the divergence of the curl is zero for all smooth vector fields, $\nabla \cdot (\nabla \times \vec{\mathbf{u}}) = 0$.

   (b) Show that for any nice f, g that $\nabla \cdot (\nabla f \times \nabla g) = 0$. Hint: you can use your results from (a).

   (c) Use index notation to show the identity for any smooth vector field $\vec{\mathbf{u}}(\vec{\mathbf{x}})$,

   $$(\nabla \times \vec{\mathbf{u}}) \times \vec{\mathbf{u}} = \vec{\mathbf{u}} \cdot \nabla \vec{\mathbf{u}} - \tfrac{1}{2}\nabla(|\vec{\mathbf{u}}|^2).$$

   (d) Use index notation to derive (starting from LHS, obtain RHS) the vector identity curl (curl $\mathbf{F}$) = grad (div $\mathbf{F}$) $- \nabla^2 \mathbf{F}$. Use this identity to show that for three-dimensional incompressible, irrotational flows, each component of the velocity is a harmonic function.

3. Beltrami flows: Let $\vec{\mathbf{u}}(\vec{\mathbf{x}})$ be a steady incompressible flow. The velocity field $\vec{\mathbf{u}}$ is called a Beltrami flow if the vorticity is everywhere parallel to the velocity. Note that parallel vectors are scalar multiples of each other, the scalar can be a function: $\vec{\boldsymbol{\omega}} = \phi(\vec{\mathbf{x}})\vec{\mathbf{u}}$.

   (a) Show that Beltrami flows are solutions of the momentum balance in the Euler equations. What is the pressure for the flow?

   (b) Determine the compatibility condition between $\phi, \vec{\mathbf{u}}$ needed to satisfy the incompressibility condition.

4. Helicity: Let $\vec{\mathbf{u}}(\vec{\mathbf{x}}, t)$ be a solution of the Euler equations on $\mathbb{R}^3$, show that the helicity $\mathcal{H}$ is a constant,

   $$\mathcal{H}(t) = \iiint_{-\infty}^{\infty} \vec{\mathbf{u}} \cdot \vec{\boldsymbol{\omega}}\, dV.$$

   Assume that $|\vec{\mathbf{u}}| \to 0$ as $|\vec{\mathbf{x}}| \to \infty$.

5. Bernoulli's theorem:

   (a) In lecture, we derived Bernoulli's theorem for solutions of Euler's equations, which roughly says *"$H = \frac{p}{\rho} + \frac{1}{2}|\vec{\mathbf{u}}|^2 + \frac{V}{\rho}$ is constant."* This understanding is a bit too "rough"; consider the two steady 2D flows (velocities and pressures):

   $$\begin{aligned} \vec{\mathbf{u}}_1 &= (-y, x) & p_1 &= \tfrac{1}{2}(x^2+y^2) \\ \vec{\mathbf{u}}_2 &= \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) & p_2 &= 3 - \frac{1}{2(x^2+y^2)} \end{aligned}$$

   By direct substitution, show that both are solutions of the steady Euler equations with no body forces and density $\rho \equiv 1$. Show that they have distinctly different H's. Explain the meaning of *"is constant"* for the versions of Bernoulli's theorem appropriate to each flow to resolve the apparent conflict. Hint: characterize each flow (what adjectives apply?).

   (b) For each flow, calculate the circulation, $\Gamma = \oint_C \vec{\mathbf{u}} \cdot d\vec{\mathbf{x}}$, on the unit circle, $C : \{|\vec{\mathbf{x}}| = 1\}$.

(Continued)

6. Polar coordinates: (use of computer-aided algebra is allowed/encouraged)

   (a) (Cylindrical) Using the fact that the Laplacian of a scalar function in polar coordinates is $\nabla^2\phi(r,\theta,z) = r^{-1}\partial_r(r\partial_r\phi) + r^{-2}\partial_{\theta\theta}\phi + \partial_{zz}\phi$, derive the $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z$ components of the Laplacian of a vector field $\vec{\mathbf{F}}$ given in polar form as $\vec{\mathbf{F}} = F_r(r,\theta,z)\hat{\mathbf{e}}_r + F_\theta(r,\theta,z)\hat{\mathbf{e}}_\theta + F_z(r,\theta,z)\hat{\mathbf{e}}_z$.

   (b) (Spherical) Consider spherical coordinates

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi.$$

      i. Find expressions for the unit vectors $\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$.

      ii. Write the velocity as

$$\vec{\mathbf{u}} = u_\rho(\rho,\theta,\phi)\hat{\mathbf{e}}_\rho + u_\theta(\rho,\theta,\phi)\hat{\mathbf{e}}_\theta + u_\phi(\rho,\theta,\phi)\hat{\mathbf{e}}_\phi,$$

      express $u_x(x,y,z,t), u_y(x,y,z,t), u_z(x,y,z,t)$ in terms of the spherical-coordinate velocity component functions. Also, find u_ρ, u_θ, u_ϕ in terms of the rectangular velocity components.