Math 575: Mathematical Fluid Dynamics Spring 2022

Homework 3 **Assigned: Fri Jan 28** **Due: Fri Feb 4**

The Stress Tensor and Boundary Conditions

0. Reading: Acheson sections 1.2 and 6.2-6.4 and the brief guide to index notation (`tensors.pdf` in the Sakai resources/summary sheets folder)

1. The Reynolds transport theorem is the Lagrangian approach to writing conservation laws on volume moving with the flow. The alternative, Eulerian approach is to examine conservation laws and fluxes on fixed *control volumes* (i.e. time independent lab-frame domains), for all choices of control volumes in the flow. Use this approach for the following on fixed (but arbitrary) volumes V: (and holds $\forall V$)

   (a) Show that the strong form of the Cauchy momentum balance equations can be obtained from
   $$\frac{d}{dt}\left(\iiint_V \rho\vec{\mathbf{u}}\,dV\right) = \iiint_V \vec{\mathbf{f}}\,dV + \iint_{\partial V} [\vec{\boldsymbol{\tau}} - \rho\vec{\mathbf{u}}(\vec{\mathbf{u}}\cdot\hat{\mathbf{n}})]\,dS.$$
   where $\vec{\boldsymbol{\tau}}$ are the surface tractions, $\vec{\boldsymbol{\tau}} = \mathbf{T}\hat{\mathbf{n}}$.

   (b) For a flow satisfying Euler's eqns, derive the rate of change of the kinetic energy, $\frac{d}{dt}\left(\iiint_V \frac{1}{2}\rho|\vec{\mathbf{u}}|^2\,dV\right)$.
   Express your answer as a surface integral of a flux over the boundary of the volume, ∂V.
   Hint: $\vec{\mathbf{u}}\cdot\nabla\vec{\mathbf{u}} = (\nabla\times\vec{\mathbf{u}})\times\vec{\mathbf{u}} + \nabla(\frac{1}{2}|\vec{\mathbf{u}}|^2)$ is an identity for all smooth $\vec{\mathbf{u}}$.

2. Basic familiarity with index notation for Cartesian tensors can be very useful for fluid dynamics, solid mechanics and many other fields. Please read the brief guide (`tensors.pdf`)[1] or other references on index notation, then work out the following:

   (a) (warm up) Let's consider the properties of the $\mathbf{A}$ tensor from the velocity gradient.
      i. If ω_k are the entries of some (arbitrary/general) vector, show that the entries $a_{ij} = \frac{1}{2}\epsilon_{ikj}\omega_k$ define an anti-symmetric 2-tensor, with $\mathbf{A}^T = -\mathbf{A}$.
      ii. Now, using that the vorticity is the curl of the velocity field, $\omega_i = \epsilon_{ijk}u_{k,j}$, show that the entries of $\mathbf{A}$ are $a_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$ starting from the equation in part (i).
      Hint: You will need an identity from the sheet and to replace some summed up indices with other convenient letters like ℓ, m, n.[2]

   (b) (main problem) The general expression for the entries in an isotropic tensor of rank 4, $\mathbb{K}$, is
   $$K_{ijk\ell} = \alpha\delta_{ij}\delta_{k\ell} + \beta\delta_{ik}\delta_{j\ell} + \gamma\delta_{i\ell}\delta_{jk},$$
   where α, β, γ are constants. The deviatoric part of the Newtonian stress tensor is given by
   $$\boldsymbol{\sigma} \equiv \lambda(\nabla\cdot\vec{\mathbf{u}})\mathbf{I} + \mu(\nabla\vec{\mathbf{u}} + \nabla\vec{\mathbf{u}}^T).$$
   Show that $\boldsymbol{\sigma} = \mathbb{K}\nabla\vec{\mathbf{u}}$ (namely $\sigma_{ij} = K_{ijk\ell}u_{k,\ell}$) if you are given two additional conditions: (1) $\boldsymbol{\sigma} = \mathbf{0}$ if $\nabla\vec{\mathbf{u}}$ is anti-symmetric, and (2) $\boldsymbol{\sigma}$ is symmetric. Identify the parameters α, β, γ in $\mathbb{K}$ in terms of the viscosity coefficients μ, λ in $\boldsymbol{\sigma}$.

3. Consider the two-dimensional flow of a compressible constant-viscosity Newtonian fluid, with $u(x,y,t)$, $v(x,y,t)$, $p(x,y,t)$, $\rho(x,y,t)$:

   (a) Write the three differential equations for the conservation of mass and conservation of momentum for a general (compressible) Newtonian fluid in the presence of gravity (acceleration g in the $-\hat{\mathbf{j}}$ direction). Expand-out the vector forms of terms to produce three scalar equations in terms of u, v, p, ρ. (Hint: Use Cauchy's equation with $\mathbf{T} = -p\mathbf{I} + \lambda(\nabla\cdot\vec{\mathbf{u}})\mathbf{I} + 2\mu\mathbf{E}$.)

[1] The only tools you will need are the definitions and identities given there.
[2] (no one should use 'o' it looks too much like zero)

(b) Consider a layer of fluid that is lying on top of a fixed, flat solid surface and is exposed to the air above. The equation for the solid surface is $y = 0$. The equation for the interface (boundary) between the layer fluid and the still and passive, inviscid air is $y = h(x, t)$ (the height of the fluid layer is the graph of a function of position and time). Assume the layer extends indefinitely in the x-direction, $-\infty < x < \infty$. Draw a sketch of the geometry of this problem and identify a level-set function $f(x, y, t)$ for the interface.

(c) Apply the three fundamental boundary conditions[3] at the fluid-solid boundary. What are the resulting equations?

(d) Write analytical expressions for the unit-tangent ($\hat{\mathbf{t}}$) and unit-outer-normal ($\hat{\mathbf{n}}$) vectors at the interface at position (x, h).

(e) Write the kinematic and no-slip boundary conditions at the fluid-gas interface.[4]

(f) Modify the stress balance boundary condition to describe the influence of *surface tension*. Surface tension is a surface force that tries to "smooth out any variations of the curvature of the boundary"; it acts in the direction perpendicular to the boundary.
Use the results from (d) to expand-out the vector equation for the surface force balance:

$$\vec{\boldsymbol{\tau}}_1 + \vec{\boldsymbol{\tau}}_2 + \sigma\kappa\hat{\mathbf{n}} = \mathbf{0},$$

where $\vec{\boldsymbol{\tau}}_1$ is the surface traction of the air on the fluid, $\vec{\boldsymbol{\tau}}_2$ is the surface traction of the fluid on the air, and the third term is the surface tension of the fluid: $\hat{\mathbf{n}}$ is the unit-outer normal to the fluid, σ is a positive constant (the surface tension coefficient, called γ in some books, like Leal), and κ is the curvature of the surface (Hint: Find the formula for the curvature of a function $y = p(x)$ from a calculus book). Assume $\mathbf{T}_1 = -p_{air}\mathbf{I}$.
Simplify as much as possible to reduce down to two scalar equations in term of p and derivatives of u, v, h.

(g) Expand-out the surface stress-balance vector equation into two scalar component equations: one by taking the dot-product of the stress equation with the normal vector to the fluid $\hat{\mathbf{n}}_2$, the other by taking the dot-product of the equation with the tangent vector $\hat{\mathbf{t}}$. (Hint: Use the orthogonality of the tangent and normal vectors, $\hat{\mathbf{t}} \cdot \hat{\mathbf{n}} = 0$.)

(h) Simplify your equations for the case of an incompressible Newtonian fluid.

[3](1) Kinematic boundary condition for the normal velocity, (2) No slip for the tangential velocity, and (3) the stress balance.

[4]The air is considered to be inviscid so there will be free slip. The condition on the normal component of velocity will not provide new information...