Math 575: Mathematical Fluid Dynamics — Spring 2022

Homework 2 | **Assigned: Fri Jan 21** | **Due: Fri Jan 28**

Conservation laws and the transport theorem

0. Reading: Acheson, Sections 6.2-6.3.

1. In one dimension, a fluid "blob" is defined by the interval $a(t) \le x \le b(t)$ with the motion of all points in the blob obeying $dx/dt = u(x,t)$. If we define the average of property f on the blob as

$$f_{\text{avg}}(t) = \frac{1}{b(t)-a(t)} \int_{a(t)}^{b(t)} f(x,t)\, dx \qquad \text{then calculate} \qquad \lim_{b\to a} \frac{df_{\text{avg}}}{dt}$$

Hint: Let $b(t) = a(t) + \epsilon h(t)$ with $\epsilon \to 0$.

2. This problem will lead you through the steps of an alternative derivation (a Lagrangian version instead of the the semi-Eulerian[1] one shown in class) of the Reynolds transport theorem:

Consider a fluid blob starting from the three-dimensional region $V(0)$ at $t = 0$. Let $V(t)$ be the region occupied by the fluid blob at time t. To describe the points in this blob label each point according to its initial position, $\vec{\mathbf{x}}(t=0) = (p,q,r)$, where $\vec{\mathbf{p}} = (p,q,r)$ is a point in the initial volume $V(0)$. These are sometimes called the "material" or Lagrangian coordinates of a point $\vec{\mathbf{x}}_{\vec{\mathbf{p}}}(t)$.

Each point in the blob satisfies the initial value problem $d\vec{\mathbf{x}}/dt = \vec{\mathbf{u}}(\vec{\mathbf{x}},t)$ with $\vec{\mathbf{x}}(0) = (p,q,r)$. Let $f(\mathbf{x},t)$ be some property that can be measured at each point in the fluid. We want to determine the rate of change of the integral of f in the moving blob, $\dfrac{d}{dt}\left(\iiint_{V(t)} f(\mathbf{x},t)\, dz\, dy\, dx\right)$.

Start by expressing the triple integral in terms of Lagrangian coordinates,

$$\iiint_{V(t)} f(\mathbf{x},t)\, dz\, dy\, dx = \iiint_{V(0)} f(\mathbf{x}(t,\mathbf{p}),t)\, J\, dr\, dq\, dp.$$

This is a change of variables from Eulerian to where $x = x(t,p,q,r)$, $y = y(t,p,q,r)$ and $z = z(t,p,q,r)$. You will need the Jacobian determinant for this change of variables,

$$J(t,p,q,r) = \left|\frac{\partial(x,y,z)}{\partial(p,q,r)}\right| \qquad \text{(in terms of } \partial_p x, \partial_q x, \cdots, \partial_r z\text{)}$$

This is a very useful change of variables because in Lagrangian form the limits of integration do not depend on time (i.e. the set of fluid particles in the blob is the same for all times). This allows you to directly interchange the time-derivative and the integration over space.

(a) The time-derivative of the integrand is a product rule (for the product fJ). You will need to take a time-derivative of the Jacobian determinant. This should work out to be

$$\frac{dJ}{dt} = [\text{something}]\, J$$

Assume this form, and determine what the "something" is. (This is called *Euler's Identity*).
Hint: When you expand J and take the time-derivative, some terms can be expressed in terms of derivatives of the velocity components (u,v,w). These velocities are Eulerian functions (functions of x,y,z), so you will have to use the chain rule in order to evaluate their partial derivatives with respect to p,q,r.

(b) Change variables back from (p,q,r) to (x,y,z) in the integral to get the RTT. *(continued)*

[1] The full Eulerian version involves fluxes in/out of a spatially-fixed control volume.

3. This problem illustrates how the convective derivative is used as the basis of the <u>*method of characteristics*</u> for solving a class of partial differential equations:

Consider following a particle moving a fluid flow, and measuring the value of function f for that particle as a function of time. The motion of the particle in the flow is given by $d\vec{\mathbf{x}}_p/dt = \vec{\mathbf{u}}(\vec{\mathbf{x}}_p, t)$. If function f is defined in terms of position and time, $f = f(\vec{\mathbf{x}}, t)$, then its value on the particle is $f_p(t) = f(\vec{\mathbf{x}}_p(t), t)$. If the rate of change (growth) of f for the particle $\vec{\mathbf{x}} = \vec{\mathbf{x}}_p(t)$ is given by a known function $g_p(t) = g(\vec{\mathbf{x}}_p(t), t)$, then f obeys the equation $\dfrac{Df}{Dt} = g$ or equivalently $\left\{ \dfrac{df_p}{dt} = g(\vec{\mathbf{x}}_p, t) \quad \text{on} \quad \dfrac{d\vec{\mathbf{x}}_p}{dt} = \vec{\mathbf{u}}(\vec{\mathbf{x}}_p, t) \right\}$

The latter are called the *characteristic equations* for the evolution of f due to flow $\vec{\mathbf{u}}$ and source function g. The particle pathlines for the flow are also called the *characteristic curves*.

(a) Consider the first-order PDE for $f(x, y, z, t)$ with a, b, c, d as given functions:

$$\frac{\partial f}{\partial t} + a(x,y,z,t)\frac{\partial f}{\partial x} + b(x,y,z,t)\frac{\partial f}{\partial y} + c(x,y,z,t)\frac{\partial f}{\partial z} = d(x,y,z,t).$$

Identify $\vec{\mathbf{u}}, g$ and write the characteristics equations for all particles in the domain.

(b) Solve the initial value problem for the first-order conservation law equation:

$$\frac{\partial f}{\partial t} + \frac{\partial(2xf)}{\partial x} + \frac{\partial(3yf)}{\partial y} = 0 \qquad f(x, y, t = 0) = x + y$$

i. Write down the three ODE's for $x_p(t), y_p(t), f_p(t)$.

ii. Solve them with initial conditions $x_p(t = 0) = x_0$, $y_p(t = 0) = y_0$, $f_p(t = 0) = f_0$.

iii. Use the initial condition to write $f_0 = x_0 + y_0$ for f_p at $t = 0$.

iv. Use the equation $x = x_p(t)$ to express x_0 as a function of x, t. Do the same for y_0.

v. Substitute these into f_p to obtain the (Eulerian) solution $f = f(x, y, t)$.

4. A one-dimensional compressible fluid blob starts at $t = 0$ with uniform density $\rho \equiv 1$ on $1 \le x \le 2$. It obeys the continuity equation $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} = 0$ with the velocity field given as $u(x, t) = e^{-x} + \dfrac{1}{1 + t}$.

(a) Find the density in blob for $t \ge 0$ as a function of position and time, $\rho = \rho(x, t)$.

(b) Find the positions of the free-boundaries, $x_1(t), x_2(t)$ (the left and right edges of the blob).

(c) Check conservation of mass by directly evaluating $\displaystyle\int_{x_1(t)}^{x_2(t)} \rho(x, t)\, dx$.

5. Use the following approach to prove that the stress tensor is symmetric ($\mathbf{T} = \mathbf{T}^T$) using the conservation of angular momentum.[2] The rate of change of angular momentum is due to the sum of the torques (force cross displacement) on points inside the volume and on its surface:

$$\frac{d}{dt}\left(\iiint_{V(t)} \vec{\mathbf{x}} \times [\rho\vec{\mathbf{u}}]\, dV \right) = \iiint_{V(t)} \vec{\mathbf{x}} \times \vec{\mathbf{f}}\, dV + \iint_{\partial V(t)} \vec{\mathbf{x}} \times \vec{\boldsymbol{\tau}}\, dS,$$

where $\vec{\mathbf{x}} = (x, y, z)$ and the surface traction is $\vec{\boldsymbol{\tau}} = \mathbf{T}\hat{\mathbf{n}}$.

Hints: (1) Expand out the product rule and make use of the statement of conservation of linear momentum from class. (2) To avoid the need for the "index notation"[3] for vectors and tensors you can expand out some products and then re-formulate the final answer in vector form. For dealing with $\vec{\mathbf{x}} \times \vec{\boldsymbol{\tau}}$, let $\hat{\mathbf{n}} = (p, q, r)$ and

$$\text{let} \qquad \mathbf{T} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \text{then write} \qquad \vec{\mathbf{x}} \times \mathbf{T}\hat{\mathbf{n}} = \mathbf{A}\hat{\mathbf{n}},$$

to find the entries in tensor $\mathbf{A}$. Then apply the tensor version of the divergence theorem to $\mathbf{A}$.

Recall that the divergence can be worked out in terms of a matrix-vector product where $\nabla = (\partial_x, \partial_y, \partial_z)$ is interpreted as a column vector, $\nabla \cdot \mathbf{A} = (\nabla^T \mathbf{A}^T)^T$. [4]

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[2] True for all "typical" fluids that dont generate internal torques (unlike ferrofluids subjected to magnetic fields).

[3] Things like $\epsilon_{ijk} x_j T_{kl} n_l$...

[4] $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ but $\nabla \cdot \mathbf{A} \ne$ "$\mathbf{A}\nabla$" doesn't work because $u\partial_x \ne \partial_x u$.