Math 575: Mathematical Fluid Dynamics — Spring 2022

Homework 1 **Assigned: Wed Jan 12** **Due: Fri Jan 21**

Flow Kinematics

kinematics: the science of pure motion, considered without reference to the matter or objects moved or the force producing or changing the motion. [i.e. $\vec{\mathbf{u}}$ is given]

– the Oxford English Dictionary

0. Reading: Acheson section 1.2 and pages 212–214 and also watch the NCFMF film on *Flow Visualization* (see link from the course web page).
1. Acheson, Page 25, Problem 1.8.
2. Consider the two-dimensional unsteady velocity field, $\left\{u(x,y,t) = \dfrac{x}{1+t},\ v(x,y,t) = -y\cos(t)\right\}$.
Note that $u > 0$ for all $x > 0$ so this is a flow going to the right. Solve $d\vec{\mathbf{x}}/dt = \vec{\mathbf{u}}$ for general $x(t), y(t)$. Use these parametric solutions to obtain the analytic expressions for the implicit curves $y = y(x)$ in each of the following cases:
   (a) The pathline for a particle starting from $\vec{\mathbf{x}}_0 = (1,1)$ at $t = 0$.
   (b) The instantaneous streamline at time $t = 0$ for a particle starting from $\vec{\mathbf{x}}_0 = (1,1)$.
   (c) The streakline at time $t = 0$ for all particles that have passed through the point $\vec{\mathbf{x}}_0 = (1,1)$ at earlier times.
   (d) Plot the curves together on a single xy graph for $1 \le x \le 15$.
3. Consider the phase plane system of linear ordinary differential equations,
$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy,$$
where a, b, c, d are real-valued constants. This system can also be written in vector form as
$$\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{D}\vec{\mathbf{x}}, \qquad \vec{\mathbf{x}} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
Interpret this ODE system as describing a two-dimensional fluid flow, $d\vec{\mathbf{x}}/dt = \vec{\mathbf{u}}$ with $\vec{\mathbf{u}}(\vec{\mathbf{x}}) = \mathbf{D}\vec{\mathbf{x}}$.
   (a) Decompose the flow into matrices for area dilations, strains, and rotations.
   (b) What is the rate of change of areas for this flow?
   (c) What is the vorticity for this flow?
   (d) Consider the two linear flow fields,
$$\mathbf{D}_1 = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \qquad \mathbf{D}_2 = \begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}$$
      i. Show that the $\mathbf{D}$'s share the same eigenvalues.
      ii. Write down the solutions $x(t), y(t)$ for each flow starting from initial condition $\vec{\mathbf{x}}(t=0) = (6,2)$.
      iii. Use (a) to describe the qualitative differences between the two flows.
      iv. Use (ii) to determine the asymptote line, $y = kx$, that is approached for $t \to \infty$.
   (e) The two following flows also share the same eigenvalues,
$$\mathbf{D}_3 = \begin{pmatrix} 2 & -1 \\ 8 & -2 \end{pmatrix} \qquad \mathbf{D}_4 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$
   How do the $x(t), y(t)$ pathlines from $(x(0), y(0)) = (1,1)$ differ in these two flows? (you may want to plot these)

4. Consider the two-dimensional steady flow described by the autonomous phase plane system:

$$\frac{dx}{dt} = 1 - \frac{x^2 - y^2}{(x^2+y^2)^2} \qquad \frac{dy}{dt} = -\frac{2xy}{(x^2+y^2)^2}$$

   (a) Show that the flow is incompressible and irrotational.

   (b) *Stagnation points* (also called equilibrium points or fixed points) are positions where the flow velocity is zero. Find the stagnation points of the flow and classify the linearized behaviors in their neighborhoods (i.e. the eigenvalues of $\mathbf{D}$ and whether the equilibrium point is a node, center, saddle, or spiral, look up phase plane analysis if needed).

   (c) Show that points starting on the unit circle, $x^2 + y^2 = 1$, remain on the circle for all times.

   (d) What information does calculating $\nabla \cdot (y^2 \vec{\mathbf{u}})$ give you about the existence of periodic solutions in the flow according to the Bendixson-Dulac theorem? (look it up)

5. Consider the velocity field,

$$u(x,y,z,t) = yz + t, \qquad v(x,y,z,t) = xz + t, \qquad w(x,y,z,t) = xy\,.$$

   Write the velocity gradient tensor. Show that this velocity field is incompressible and irrotational for all times.

6. Derive the general expression for the local instantaneous planar area strain rate in a 3D flow $\vec{\mathbf{u}} = \vec{\mathbf{u}}(\vec{\mathbf{x}})$ using the following steps:
   Consider three points moving with the flow, $\vec{\mathbf{x}}_1(t)$, $\vec{\mathbf{x}}_2(t)$, and $\vec{\mathbf{x}}_3(t)$. Let these points be adjacent corners of a parallelogram with edge vectors, $\vec{\mathbf{p}} = \vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2$ and $\vec{\mathbf{q}} = \vec{\mathbf{x}}_3 - \vec{\mathbf{x}}_2$. Recall that the area of a parallelogram with edges $\vec{\mathbf{p}}, \vec{\mathbf{q}}$ is given by the length of the cross product vector, Area $= A(t) = |\vec{\mathbf{p}} \times \vec{\mathbf{q}}| = |\vec{\mathbf{p}}||\vec{\mathbf{q}}| \sin\theta$, where θ is the angle between the edge vectors.

   (a) Evaluate $\dfrac{1}{A}\dfrac{dA}{dt}$. Your answer will be in terms of $\vec{\mathbf{p}}, \vec{\mathbf{q}}, A$ and $\vec{\mathbf{u}}_k$ at $\vec{\mathbf{x}}_k$ for $k = 1, 2, 3$.

   Hint: Some miscellaneous vector algebra/calculus:

$$|\vec{\mathbf{a}}| = (\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})^{1/2} \qquad \vec{\mathbf{a}} = |\vec{\mathbf{a}}|\hat{\mathbf{a}} \qquad \hat{\mathbf{a}} = \vec{\mathbf{a}}/|\vec{\mathbf{a}}| \ \text{(unit direction vector)} \qquad \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta$$

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\theta \qquad \vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}} \qquad (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot (\vec{\mathbf{c}} \times \vec{\mathbf{d}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{d}}) - (\vec{\mathbf{a}} \cdot \vec{\mathbf{d}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})$$

$$\frac{d}{dt}\left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right) = \frac{d\vec{\mathbf{a}}}{dt} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \frac{d\vec{\mathbf{b}}}{dt} \qquad \text{(and other versions of the product rule)}$$

$$\frac{d}{dt} f(\vec{\mathbf{a}}(t)) = \nabla f(\vec{\mathbf{a}}) \cdot \frac{d\vec{\mathbf{a}}}{dt} \qquad \text{(and other versions of the chain rule)}$$

   (b) Write the lengths of the edges as $h = |\vec{\mathbf{p}}|$ and $k = |\vec{\mathbf{q}}|$. The edge vectors can then be written in terms of unit direction vectors as $\vec{\mathbf{p}} = h\hat{\mathbf{p}}$, $\vec{\mathbf{q}} = k\hat{\mathbf{q}}$. Take the limit of (a) as $(h,k) \to 0$ with $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ held fixed. Simplify the answer as much as possible. Your answer will be in terms of $\hat{\mathbf{p}}, \hat{\mathbf{q}}$, and $\nabla\vec{\mathbf{u}}(\vec{\mathbf{x}}_2)$. Hint:

$$D_{\hat{\mathbf{a}}} f(\vec{\mathbf{x}}_0) = \lim_{h \to 0} \frac{f(\vec{\mathbf{x}}_0 + h\hat{\mathbf{a}}) - f(\vec{\mathbf{x}}_0)}{h} \qquad \text{(the limit definition of the derivative)}$$

$$D_{\hat{\mathbf{a}}} f(\vec{\mathbf{x}}) = \nabla f \cdot \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} \qquad \text{(the directional derivative)}$$

   (c) Show that for a 2D flow, $\vec{\mathbf{u}} = (u(x,y), v(x,y), 0)$, with $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3$ in the xy plane, part (b) yields the divergence, $\nabla \cdot \vec{\mathbf{u}} = u_x + v_y$, as the local dilation rate.[1] Hint: If you can't find a better geometric reduction, you can plug-in $\hat{\mathbf{p}} = (1,0,0)^T$ and $\hat{\mathbf{q}} = (\cos\theta, \sin\theta, 0)^T$.

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[1] Given how complicated this "2D" problem is, you can imagine that this approach to calculating the 3D problem for the volumetric strain rate of a parallelepiped, Volume $= V(t) = |\vec{\mathbf{r}} \cdot (\vec{\mathbf{p}} \times \vec{\mathbf{q}})|$, would be MUCH more laborious.