Math 553: Asymptotics and Perturbation Methods Fall 2022

Problem Set 8 **Assigned Fri Nov 11** **Due Fri Nov 18**

Matched asymptotic expansions and boundary layer problems

0. Reading: B&O Chapter 9 (primarily sections 9.1-9.3).

1. Consider the problem for $y(x)$ on $0 \le x \le 1$ with $\epsilon \to 0$:

$$\epsilon^3 y'' - y' + 10\,\epsilon\, e^{2x} = 0 \qquad y(0) = 7\epsilon^2 \qquad y(1) = 3\epsilon\,.$$

This problem is about the scaling from the ϵ^β factors, so let me simplify things a little by saying the solution has a boundary layer (only) at $x_* = 1$.

   (a) Determine the leading order outer solution.
   (b) Determine the leading order inner solution.
   (c) Determine the boundary layer correction and write the uniformly valid solution.

2. Consider the differential equation

$$\frac{d^4 y}{dx^4} + x\frac{dy}{dx} - y = 0 \qquad \text{on } 0 \le x \le L$$

in the limit $L \to \infty$:

   (a) Let $y(x) = u(z)$ with $x = Lz$ to write the problem on the domain $0 \le z \le 1$ as a singular perturbation problem for $u(z)$ in terms of $\epsilon = 1/L \to 0$.

For the following two sets of boundary conditions, construct a uniformly valid solution on the problem using the following steps: (i) Show that the entire expansion of the outer solution is given by the leading order term, (ii) Determine the first two terms in the expansion of the inner solution, $U(Z) \sim U_0(Z) + \delta U_1(Z)$, (iii) Use higher-order matching to confirm the matching of the inner and outer solutions, (iv) Write the uniformly valid solution as the sum of the outer and inner solutions minus the overlap terms from matching. (Maple/Mathematica are recommended since $U_1(Z)$ has several terms from the homogeneous and particular solutions.)

   (b) $u(0) = 0 \qquad u'(0) = 1 \qquad u'(1) = 0 \qquad u''(1) = 0.$
   This is sometimes called a *"corner layer"* because the boundary layer does not change the value of the solution to leading order, but there is a dramatic change with respect to the derivative of u.
   (c) $u(0) = 0 \qquad u'(0) = 1 \qquad u(1) = 0 \qquad u'(1) = 0.$
   Using the leading order uniform solution it might look like $u'(1) = 0$ is not being satisfied, but this is resolved by including $U_1(Z)$.

3. Use matched asymptotics to determine the leading-order representation of the unique solution on $0 \le x \le 1$ of

$$\epsilon y'' - y' + (y')^3 = 0, \qquad \epsilon \to 0$$
$$y(0) = 0, \qquad y(1) = 1/2$$

Determine all of the possible distinguished limited and layer positions x_* and show that the solution must be unique.

(continued)

4. Find the leading-order uniformly-valid asymptotic solution on $0 \le x \le 1$ of

$$\epsilon^2 y'' + (x^2 + 2x^3)y' - (2x^2 + \sqrt{\epsilon})y = 0, \qquad \epsilon \to 0$$

$$y(0) = 1, \qquad y(1) = 1$$

This problem has an outer solution ($\alpha = 0$) and two different distinguished limits giving a boundary layer within a boundary layer (an inner solution and an inner-inner solution) (both with $\alpha > 0$), this is called a *"triple deck"* problem. Hint: for the matching and application of the boundary condition: apply the boundary condition to the inner-inner solution, match that solution to the inner solution, then match the inner solution to the outer solution.

5. (Optional, extra credit) Burgers' equation is a classic nonlinear partial differential equation that comes up in many contexts in fluid dynamics and related models:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon^2 \frac{\partial^2 u}{\partial x^2} \qquad u(x, t = 0) = f(x)$$

If the initial condition $f(x)$ is finite and bounded in amplitude, then the solution $u(x, t)$ will have that property too for all times. It will evolve slowly ($\partial_t u = O(1)$), but under some conditions, the solution can develop steep spatial gradients. For $\epsilon = 0$ the equation for the leading order outer solution can be solved using the "method of characteristics", but lets not do that, all we need is:

(a) Assume the solution develops an interior boundary layer at some position (which may be moving with time), call it $x_*(t)$. Assume that the inner solution for $\epsilon \to 0$ can written in the form

$$u(x, t) = U\left(\frac{x - x_*(t)}{\epsilon^\alpha}, t\right)$$

Write the PDE for $U(X, t)$.

(b) Assume $U(X, t) \sim U_0(X)$, determine α and if $dx_*/dt = c$ (constant speed), determine the first order ODE for $U_0(X)$. (You do not need to solve this ODE!)

(c) For asymptotic matching to the outer solutions to the left and to the right of x_*, U_0 will have to approach constants, call these:

$$\lim_{x \to x_*^-} u_{out} = L \qquad \lim_{x \to x_*^+} u_{out} = R$$

Determine the two constants in the U_0 ODE in terms of L, R.

(d) Show that a U_0 inner layer is not possible if $L < R$.
(Hint: Show this yields a contradiction.)