Math 553: Asymptotics and Perturbation Methods Fall 2022

Problem Set 5 **Assigned Thurs Oct 13** **Due Fri Oct 21**

The method of steepest descents for integrals

0. Reading: Bender and Orszag, section 6.6

1. Recall the definition of the Airy function given in class as a complex contour integral,

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_C e^{xt - t^3/3}\, dt$$

where C was any contour starting from the descent region in quadrant III and ending in the descent region in quadrant II.
Find the **first three** non-trivial terms in the expansion of $\mathrm{Ai}(x)$ as $x \to 0$[1] using the following steps:

   (a) Write $\mathrm{Ai}(x)$ with C constructed from the rays $\theta = 2\pi/3$ and $\theta = 4\pi/3$.
   (b) Write the integrals for the n^{th} derivative of $\mathrm{Ai}(x)$ at $x = 0$ for general $n = 0, 1, 2, \cdots$ (Taylor series coefficients)
   (c) Write the first three non-zero terms using only $\Gamma(2/3)$ and other constants.
   Hint: $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and see page 569.

2. For the linearized KdV equation for dispersive water waves,

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad -\infty < x < \infty$$

   (a) If the Fourier transform solution of this PDE with initial condition $u(x, t = 0) = ce^{-b|x|}$ with $b > 0$ is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+k^2} e^{ikx - ik^3 t}\, dk$$

   then what are the constants b, c?
   (b) Find the leading order asymptotics of the solution as (i) $x \to +\infty$ and (ii) $x \to -\infty$. Draw all contours used and show all work needed to justify your answers.
   Hint: There will be contributions from residues of pole singularities.
   (c) The results from (b) are good for large $|x|$, but they are not valid near the origin ($x \to 0$); the solution does not really blow-up. What is the result for $u(0, t)$ you can derive for $t \to \infty$ using the method of stationary phase? [2] This breaks-down too, what is the value of $u(0, 0)$?

3. Consider the differential equation for $y(x)$ on $0 \le x < \infty$:

$$\frac{d^4 y}{dx^4} = y + \frac{x}{4}\frac{dy}{dx}$$

   (a) Express the solutions in terms of a generalized Laplace transform integral: $y(x) = \int_C Y(t)e^{xt}\, dt$. Find $Y(t)$ and determine the descent regions to eliminate boundary terms.
   (b) Pick contours to provide **four** linearly independent solutions for this fourth-order ODE.
   (c) Apply the method of steepest descents to obtain the leading order behavior for each of the four solutions as $x \to +\infty$. Hint: One grows, three decay as $x \to \infty$.

[1] This is not a steepest descents problem (it is mostly a HW#2 problem), but it uses some of the same elements: choice of contours, parametrizing contour integrals and Laplace-type integrals.

[2] Often PDE solutions are studied in the limit $(x, t) \to \infty$ with $x = vt$ for fixed $v = O(1)$ with different cases for v in $-\infty < v < \infty$ in the limit of $t \to \infty$.