Math 553: Asymptotics and Perturbation Methods Fall 2022

Problem Set 4 **Assigned Fri Sep 30** **Due Fri Oct 7**

Stationary Phase

0. Reading: Bender and Orszag, section 6.5.

1. Bender and Orszag, page 312, problem 6.56 (all).

2. Bender and Orszag, page 312, problem 6.54 (b).

3. (OPTIONAL, extra credit) Asymptotics of integrals is often used to help understand the behavior of solutions to PDE problems.[1] The formula[2]

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)} \right] f(y)\, dy$$

gives the solution of the Dirichlet initial-boundary value problem for the heat equation on $0 \le x < \infty$ and $t \ge 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = 0 \qquad u(x,0) = f(x)$$

If the initial condition function, $f(x)$, does not satisfy the boundary condition at the origin ($u(0,t) = 0$), i.e. $f(0) \neq 0$, then the solution $u(x,t)$ must work out what to do with the discontinuity between these conditions. If you plug-in $x = 0$ into the above integral, it reduces to $u(0,t) = 0$ for all times. So at $x = 0$ the initial condition $u(0,0) = f(0)$ is not really satisfied... Lets consider the specific case $f(x) = e^{-x}$.

   (a) For $x > 0$ the solution u will be positive and $u \to 0$ for $x \to \infty$, with $u = 0$ at $x = 0$. There will be a maximum of u at some x_* at each time. Find $x_*(t)$ for $t \to 0$.

   (b) What is the limit of $u(x,t)$ for $t \to \infty$? The long-time behavior should approach a "dipole similarity solution".

---

[1] This is not a stationary phase problem, it could have gone on one of the previous problem sets...

[2] The formula can be obtained from the method of images