Math 553: Asymptotics and Perturbation Methods Fall 2022

Problem Set 2 **Assigned Fri Sep 16** **Due Fri Sep 23**

Basic methods for Asymptotics of Integrals

0. Reading : Bender and Orszag, sections 6.1–6.3 pages 247–261. (Hinch section 3.4 covers nonlocalized integrals)

1. Bender and Orszag, page 307, problem 6.7a,b,c,d,h.
   For each integral, if the AE starts with a constant, continue to next order to find the first term with non-trivial dependence on x.
   For (d), express the leading constant in terms of an exponential integral, see page 575.

2. Bender and Orszag, page 308, problem 6.17a.

3. Bender and Orszag, page 308, problem 6.18b
   Either use integration by parts, or change to an integral on $\int_x^\infty$ via the change of variables $u = xt$ and then introduce a δ breakpoint, or see page 252.

4. Consider the nonlinear ordinary differential equation

$$\frac{d^2u}{dx^2} = u - \tfrac{3}{2}u^2. \tag{1}$$

   In terms of phase plane analysis, this problem has a center point at $u = 2/3$ and and a saddle point at $u = 0$. Surrounding the center point, there is a continuous family of periodic solutions $u(x)$ with minima covering the range $0 \le u_{\min} \le 2/3$. Multiplying equation (1) across by du/dx and integrating each term $\int (\cdot)\, dx$, we can obtain the first-order equation (called the first integral),

$$\left(\frac{du}{dx}\right)^2 = u^2 - u^3 + C \tag{2}$$

   where C is a constant of integration.

   (a) If we define ϵ by $u_{\min} = \epsilon > 0$, use the first integral to show that the corresponding maximum of the periodic solution is

$$u_{\max} = \tfrac{1}{2}\left(1 - \epsilon + \sqrt{1 + 2\epsilon - 3\epsilon^2}\right)$$

   Hint: If the local minimum of $u(x)$ is $u = \epsilon$, what is the value for C in (2)?

   (b) Using (2), the period of oscillation for the periodic solutions can be written as

$$L = 2\int_{u=u_{\min}}^{u=u_{\max}} dx = 2\int_{u_{\min}}^{u_{\max}} \frac{du}{du/dx} = 2\int_{\epsilon}^{u_{\max}(\epsilon)} \frac{du}{\sqrt{u^2 - u^3 - \epsilon^2 + \epsilon^3}} = L(\epsilon)$$

   Find the leading order behavior of $L(\epsilon)$ for $\epsilon \to 0$.

   Hint: Start with a change of variables to shift $u = \epsilon + t$.