Math 553: Asymptotics and Perturbation Methods Fall 2022

Problem Set 1 **Assigned Wed Sep 7** **Due Fri Sep 16**

Introduction and algebraic equations

-2. Homework policy: Homework is to be submitted via `www.gradescope.com`.

You may discuss progress on problems with other students, but your final work must be written up independently. Show enough steps for me to be able to follow your solution process.
Unexcused late homeworks will not be accepted. Any extensions or excuses must be requested before the due date.

-1. (Computer-aided algebra (CAA) policy) Use of a computer algebra program (`Maple`, `Mathematica`, etc) is recommended for solving some of the more algebraically intensive problems on various homeworks. If you work out problems with computer-aided algebra, you must turn in a copy of the worksheet you used to generate your solution (pdf of print-out of commands/results).

0. Reading: Bender and Orszag is not the ideal book for this introductory background, but here are some relevant sections: pp.317–325, 123–127. If you'd like further background, see Hinch (Chapters 1, 2) or Miller (Chapter 1) or Holmes (Chapter 1).

1. Powers of ϵ are gauge functions in many problems, $\delta_n = \epsilon^n$. In some problems "*exponentially small terms*" (e.s.t.) also arise, like $\delta_* = e^{-1/\epsilon}$ for $\epsilon \to 0^+$. The fact that EST's are smaller than all powers, $\delta_* \ll \delta_n$, is an important result used in many problems.
   (a) Try proving this using $\lim_{\epsilon\to 0^+} \delta_*(\epsilon)/\delta_n(\epsilon)$ via L'Hopital's rule. Does this work?
   (b) Show that you can work this out using: (a) $\epsilon^n = e^{n\ln\epsilon}$, and (b) $\epsilon = 1/\lambda$ with $\lambda \to \infty$.
   (c) On the other hand, logarithms are "weakly large": show that $1 \ll \ln(\lambda) \ll \lambda^n$ for any $n > 0$ for $\lambda \to \infty$ (so $|\ln(\epsilon)| = \ln(1/\epsilon) \ll \epsilon^{-n}$ for any $n > 0$).

2. Consider the behavior of the function defined by the integral $f(x) = \displaystyle\int_0^\infty \frac{e^{-2t}}{1+3xt}\,dt$ for $x \to 0$.
   (a) Calculate the n^{th} derivative at $x = 0$, $f^{(n)}(0)$, for $n = 0, 1, 2, \cdots$ (general case).
   Hint: Leibniz's rule, $\int_0^\infty t^k e^{-st}\,dt = k!/s^{k+1}$, and the geometric series $1/(1-w) = \sum_{n=0}^\infty w^n$.
   (b) Write the Taylor series for $f(x)$ expanded at $x = 0$. Show that this is a divergent series with the radius of convergence being zero.
   (c) A finite radius of convergence is associated with the occurrence of a singularity of the function limiting the radius of the largest disk of analytic behavior that could be drawn around the expansion point. $f(x)$ is finite for $x \geq 0$; explain why is the integral singular for all negative x.
   (d) Let $S_N(x) = \sum_{n=0}^N f_n x^n$ be the finite-truncation of the Taylor series. Let $|f_{N+1}x^{N+1}| = |f_N x^N|$ be the truncation condition to determine $N = N_*(x)$ at a given x value.
   (e) For $x = 1/7$, compute a plot $|S_N(x) - f(x)|$ vs. N for $N = 1, 2, \cdots, 8$ to show that N_* is close to the optimal asymptotic truncation. (Hint: Use Maple or etc for the numerical value of $f(1/7)$.)

3. Let $g(y) = \exp(-(x-y)^2/(4t))$ with x, t being fixed constants. Determine the first four non-zero terms in the expansion of g for $y \to 0$. Let $x = \eta\sqrt{4t}$ to re-write the answer in terms of (η, y, t).

   The above expansion is one part of the justification to show that the solution of the initial-value problem for the heat equation, $u(x,t) = \int_{-\infty}^\infty f(y)e^{-(x-y)^2/(4t)}\,dy/\sqrt{4\pi t}$, can be written as an asymptotic expansion for $t \to \infty$ in the form:

$$u \sim \frac{e^{-\eta^2}}{\sqrt{4\pi t}}\left[\int_{-\infty}^\infty f(y)\,dy + \frac{\eta}{t^{1/2}}\int_{-\infty}^\infty y f(y)\,dy + \frac{2\eta^2-1}{4t}\int_{-\infty}^\infty y^2 f(y)\,dy + \frac{h_3(\eta)}{t^{3/2}}\int_{-\infty}^\infty y^3 f(y)\,dy + \cdots\right]$$

   Determine the $\boxed{h_3(\eta)}$ coefficient function.

(continued)

4. Consider the equation $(x-3/\epsilon)^4 = 27x$ in the limit $\epsilon \to 0$.

   (a) Solve by iteration to determine first three terms in the expansion of the positive real-valued solution $x \sim \delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2$. Hint: You can determine δ_0 without expanding the quartic term.

   (b) To consider the accuracy of your solutions, how many terms in the expansion must you keep so that the error vanishes in the limit, $|x(\epsilon) - x_{\mathrm{AE}}| \to 0$ as $\epsilon \to 0$. Comment on the accuracy of the leading order estimate $x \sim \delta_0 x_0$.

5. Consider the algebraic equation $\epsilon^{14}x^4 - 6\epsilon^5x^3 - 24\epsilon^2x^2 + 30\epsilon^3x + 120 = 0$ in the limit $\epsilon \to 0$.

   (a) Show that there are no regular solutions.

   (b) Determine the leading order nontrivial term in the expansion of each of the four solutions.

6. Consider the system of equations for $\epsilon \to 0$: $\begin{cases} \epsilon x - 4y = 1 \\ \epsilon^2 x + y = 2 \end{cases}$

   This system has a unique solution for every $\epsilon > 0$, but DO NOT solve it directly. Explain why setting $\epsilon = 0$ does not lead to an acceptable leading order solution. Rescale the solution as $x = \delta(\epsilon)X(\epsilon)$ and $y = \sigma(\epsilon)Y(\epsilon)$ with $X, Y = O(1)$. Determine the dominant balance. Note that for systems of equations, the dominant balance may occur within a single equation with the other terms and equations being subdominant. Determine the first two terms in the expansions of $x(\epsilon), y(\epsilon)$.

7. Consider the matrix $\mathbf{A}(\epsilon) = \begin{pmatrix} e^{-3\epsilon} & -5+3\epsilon \\ -2-7\epsilon & -2\cos(8\epsilon) \end{pmatrix}$.

   For $\epsilon \to 0$, use matrix perturbation theory to find the first two terms in the expansion of each eigenvalue. (DO NOT try to expand the determinant $|\mathbf{A}(\epsilon) - \lambda\mathbf{I}| = 0$)

8. (OPTIONAL, Implicit asymptotics for large solutions) Separation of variables for solutions to the heat equation subject to Robin boundary conditions produces equations for the eigenvalues in the form

$$\tan(\lambda) = -\lambda$$

   Introduce an "artificial perturbation parameter" ϵ and obtain the first three terms in the expansion of λ for $\lambda \to \infty$. To solve $\sin(\lambda)/\lambda = -\cos(\lambda)$, consider $\epsilon\sin(\lambda)/\lambda = -\cos(\lambda)$ with $\epsilon \to 0$ and show your terms to be ordered, $\lambda_0 \gg \lambda_1 \gg \lambda_2$, with respect to some limiting parameter, even when you restore $\epsilon = 1$.
   Reading B+O pages 319–321, 324-325, and problem 7.8 on page 361 may be helpful.
   Note that the numerical values for the series of λ starts with $\lambda = \{2.0287, 4.9131, 7.9786, \cdots\}$ – the asymptotic approximation does very well even when λ isn't very large.
   Hint: You should be able to work out λ_0, λ_1 by hand, but you may appreciate CAA for getting λ_2.