# Math 551: Applied PDE and Complex Vars — Fall 2023

**Problem Set 7** — **Assigned Fri Oct 20** — **Due Fri Oct 27**

**Separation of Variables and Eigenfunction expansions for PDE BVP (Part 2)**

0. Reading: Lecture notes 13-18 and
   - Haberman, **Section 8.6** and Section 2.5.1.
   - Haberman, Sections 7.3, 7.2, 7.4, 7.5
   - Take a look at the Wikipedia article on "Hearing the shape of a drum" – This discusses how the eigenvalues (which give the natural frequencies of vibration, squared, $\lambda = \omega^2$) of a Helmholtz equation derived from separation of variables for a wave equation, $u_{tt} = \nabla^2 u$, with homogeneous Dirichlet boundary conditions are related to the shape of the domain.

1. Haberman, page 371, Problem 8.6.1b. **(Version 1)**[1]
   Follow Haberman's instructions: "Do not reduce to homogeneous BC's."
   Namely, use the following steps to carry out the "one-dimensional eigenfunction expansion approach":
   (a) Use the spatial direction ($x$ or $y$) with homogeneous BC's to pick the direction and variable for the 1-D eigenfunctions, then write the solution as a single-summation eigen-expansion in terms of these eigenfunctions times coefficient functions depending on the other variable.
   (b) Determine the ODE BVP's for each of the coefficient functions. Solve each of those BVP's in terms of a second eigen-expansion.
   Hint: If you are confused by the inhom BC, consider what you'd do with BC $u(L, y) = h(y)$.
   (c) Plugging the expansions for each of the coefficients back into your solution from (a), your final answer will be a double-sum with $u(x, y) = \sum_n \sum_m c_{nm} f_n(x) g_m(y)$.
   Your answer should have an explicit formula for the constants with everything worked-out except for a double integral of $Q(x, y)$.

2. Haberman, page 371, Problem 8.6.1b. **(Version 2)**[2]
   This time ignore Haberman's instructions about "Do not reduce to homogeneous BC's".
   Instead, use the following steps to carry out the "two-dimensional eigenfunction approach":
   (a) Homogenize the PDE and BC's. Find the two-dimensional eigenfunctions $\phi(x, y)$ for the PDE eigenvalue problem (solve the Helmholtz problem, $\nabla^2\phi = -\lambda\phi$, using separation of variables).
   (b) Return to the original full problem, $\nabla^2 u = Q$ with BC's, and project onto the adjoint eigenfunctions $\psi_k$, $\langle \nabla^2 u, \psi_k\rangle = \langle Q, \psi_k\rangle$, to get equations for the constant coefficients in the expansion $u = \sum_k c_k \phi_k$.
   Your final answer will be a double-sum with $u(x, y) = \sum_n \sum_m c_{nm} f_n(x) g_m(y)$.
   Your answer should have an explicit formula for the constants with everything worked-out except for a double integral of $Q(x, y)$.

   To do the projection, here is an improved version of the 2-D version of Green's formula for linear self-adjoint second order PDE operators (like Q 4 below) (see eqn 7.5.7 in Haberman, page 289),

$$\langle v, \mathrm{L}u\rangle_2 = \boxed{\iint_D v[\nabla\cdot(p\nabla u) + qu]\, dA = \oint_C p\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \iint_D u[\nabla\cdot(p\nabla v) + qv]\, dA}$$

   If $p \equiv 1$ and $q \equiv 0$, then this reduces back to (7.5.7) – and consider it with $v = \phi$:

$$\langle \phi, \mathrm{L}u\rangle = \iint_D \phi\nabla^2 u\, dA = \oint_C \left(\phi\frac{\partial u}{\partial n} - u\frac{\partial \phi}{\partial n}\right) ds + \iint_D u\nabla^2\phi\, dA.$$

*(continued)*

[1] There is a solution for this problem listed in the back of the textbook, but it has a typo :(

[2] There could have been a question about "Version 3" of the solution, done via super-superposition using the sum of the solution of the Laplace equation with the inhomogeneous BC and the Poisson solution with all-homogeneous BC's, but I decided two versions were enough for this HW, the HW solutions will show Version 3 too.

The line integral around the boundary of the domain needs to be broken up into integrals along each part of the boundary. I hope you'll find this approach to be an easy-to-use alternative that gives you the same solution as Problem 1 (i.e. the 1-D Version 1 approach), but much faster.

3. Haberman, page 372, Problem 8.6.9.

   (a) For Part (a), DO NOT use "physical reasoning" (What is physical reasoning?), instead derive the solvability condition from the Fredholm alternative theorem. Hint: See the notes from Lecture 18.[3]

   (b) Use the "2-D eigenfunction approach" to construct the solution for Part (b).

   (c) For Part (c), the solution of the heat equation can be written as

   $$u(x,y,t) = \sum_n \sum_m b_{nm}(t)\phi_{mn}(x,y).$$

   The $b_{mn}$ coefficients will satisfy inhomogeneous ODE's in time with IC's set by the IC for $u(x,y,0)$. These coefficients should show exponential decays to the constants for the solution of the original Poisson equation.
   Explain which coefficient has the "arbitrary constant".
   Explain why this constant can be found the same way even if the domain has any shape in the $xy$ plane (see problem 8.6.10). (Hint: what is the adjoint eigenfunction for $\lambda_0 = 0$?)

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4. (Optional, extra credit) Haberman, page 295, problem 7.6.3. Assume that $p(x,y,z)$ is a given positive function.
   This problem basically asks you to combine the steps that were used in the Rayleigh quotient proof from the notes from Lecture 6 for the Sturm-Liouville ODE operator $\mathrm{L}u \equiv (p(x)u')'$ and generalize the PDE operator to be $\mathrm{L}u \equiv \nabla \cdot (p(\mathbf{x})\nabla u)$ instead of the Laplace operator $\mathrm{L}u \equiv \nabla^2 u$ (see the Rayleigh quotient proof done from the Lecture 17 notes for the Helmholtz eigenvalue equation).

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[3] I'm using $S(x,y)$ rather than Haberman's notation $Q(x,y)$ for source terms because when I write notes by hand, I think my $Q$ will get confused with $\phi$'s because they might look very similar in my sloppy writing.