Math 551: Applied PDE and Complex Vars — Fall 2023

Problem Set 6 — **Assigned Mon Oct 9** — **Due Sat Oct 21**

Eigenfunction expansions for inhomogeneous PDE IBVP (Part 1)

0. **Readings from Haberman**: Chapter 2 is a very good introduction with good basic examples. Chapter 8 goes into more details on more complicated problems (with inhom. BC's and forcing fcns).

   - Heat equation: Sections 2.3, 2.4.1, Section 8.4
   - Wave equation: Section 8.5 (part 1: pp. 358–360).
   - Poisson's/Laplace's equation: Section 2.5.1. Section 8.6 (part 1: pp. 366–369).

   and Lectures 11-15.

1. Consider the problem for the forced heat equation on $0 \leq x \leq \pi$ with $t \geq 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6xe^{-5t}, \tag{1a}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 5\sin(t), \qquad \left.\frac{\partial u}{\partial x}\right|_{x=\pi} = 7e^{3t} \qquad \text{for } t > 0 \tag{1b}$$

$$u(t=0) = 9e^{-2x} \qquad \text{for } 0 \leq x \leq \pi. \tag{1c}$$

   Consider the solution of this problem as an eigenfunction expansion in the form $u(x,t) = \sum_k b_k(t)\phi_k(x)$:

   (a) Identify the eigenfunctions $\phi_k(x)$ and eigenvalues/separation constants λ_k.

   (b) Identify all integrals that need to be evaluated in this problem. Work out the $||\phi_k||^2$ integrals and concretely write out all of the others so they could be evaluated by Maple or Mathematica, but DO NOT work out these integrals.

   (c) Write the ODE's and IC's satisfied by the $b_k(t)$ coefficient functions. Concretely write out all parts of the problem so they could be evaluated by a computer algebra program, but DO NOT solve these ODE problems.

   (d) Write a one or two sentence response to the statement: *"Since the eigenfunctions selected in part (a) satisfy homogeneous BC's, the expansion for the solution does not satisfy the original boundary conditions (or the initial conditions at the edges)."*

   (e) Expand out the $k = 0$ ODE problem from (c) and calculate the solution for $b_0(t)$.
   From (a), this problem has a zero eigenvalue, $\lambda_0 = 0$. Explain if this presents trouble regarding existence or uniqueness of the solution $u(x,t)$.

2. Consider the problem on $0 \leq x \leq \pi$ for the forced/damped wave equation:

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 9t^3, \tag{2a}$$

$$u(x=0) = t, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=\pi} = 5, \tag{2b}$$

$$u(t=0) = 4x, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 4\cos(2x). \tag{2c}$$

   (a) Write down the ODE IVP's for the $b_k(t)$ coefficient functions in the appropriate eigenfunction expansion solution, $u(x,t) = \sum_k b_k(t)\phi_k(x)$. Like Problem 1, DO NOT solve these ODE problems.

   (b) Consider the homogeneous versions of your $b_k(t)$ ODE's (zero-out the RHS forcing from BC's and source terms). Solve these ODE's for the general $b_k(t)$ (dont use the IC's). Are there are modes that behave differently than the rest?

(continued)

3. A comparison of "x-BVP" vs. "y-BVP" eigenfunction expansions for a Laplace equation problem for $u(x, y)$ with Dirichlet and Robin boundary conditions. Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 \le x \le \ell \qquad 0 \le y \le h \tag{3a}$$

$$u(0, y) = 0 \qquad u(\ell, y) = 0 \qquad 0 \le y \le h \tag{3b}$$

$$u(x, 0) = 0 \qquad u_y(x, h) + 3u(x, h) = f(x) \qquad 0 \le x \le \ell \tag{3c}$$

   (a) The "x-BVP" eigenfunction expansion takes advantage of the homogeneous left/right BC's. Show that the solution can be written in the form

$$u(x, y) = \sum_{k=1}^{\infty} d_k \sinh\left(\frac{k\pi y}{\ell}\right) \sin\left(\frac{k\pi x}{\ell}\right). \tag{4}$$

   and determine the equations for the d_k constants.

   (b) Using the y-direction to determine the set of eigenfunctions yields the solution in the form

$$u(x, y) = \sum_{k=1}^{\infty} b_k(x) \sin(\sqrt{\lambda_k}\, y). \tag{5}$$

      i. Homogenize the BC's to determine the equation for the eigenvalues λ_k. (DO NOT try to solve this equation.) Hint: see Haberman Chapter 5.8.

      ii. Assuming you are given all of the λ_k's, determine the ODE BVP for the $b_k(x)$ coefficient functions by integrating the projection of problem onto the eigenfunctions (as usual):

$$\langle \text{PDE}, \phi_k \rangle \qquad \to \qquad \int_0^h \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \phi_k(y)\, dy = 0 \qquad \text{and so on} \cdots \tag{6}$$

4. Recall Problem 4 from Homework 3 on reducing Gibbs' phenomenon by separating out inhomogeneous boundary conditions.
   A similar approach can be applied to many PDE problems. This is done by writing the solution in two parts as $\boxed{u(x, t) = u_B(x, t) + u_F(x, t)}$ where:

   - The "boundary solution" u_B satisfies the original inhomogeneous boundary conditions, but with a modified forcing/source term $m(x, t)$ and a "quasi-steady" (meaning no time-derivative terms) version of the PDE, $\boxed{Lu_B = m(x, t)}$ We select forcing fcn m to get an "easy" soln for u_B.
   - The "forced solution" u_F satisfies the full PDE with homogeneous BC's (and modified source term, $S - m$). Soln u_F can be written as an eigenfunction expansion with weaker Gibbs phenomenon at the boundaries.

   The PDE problem for $u_F(x, t)$ will be modified from the original problem depending on the form of $u_B(x, t)$. This is determined by substituting-in $u = u_F + u_B$ into the original problem after you've determined $u_B(x, t)$ (see Haberman, Chapter 8.2 for a similar discussion).
   For Problems 1 and 2 from this Homework, the spatial operator is $Lu \equiv \partial^2 u/\partial x^2$.
   (Note: You can do this problem without having solved Problems 1 or 2 first.)

   (a) For Problem 2, let $m = 0$ and determine $u_b(x, t)$, and then write out the PDE problem satisfied by u_F, namely the PDE with inhomogeneous forcing term, IC's, and BC's. (DO NOT solve it)

   (b) For Problem 1 with $m = 0$, what happens when you try to determine $u_B(x, t)$?

   (c) For Problem 1, set $u_B(x, t) = C(t)x + D(t)x^2$. Determine C, D to satisfy the boundary conditions, then write the full PDE problem for $u_F(x, t)$. (DO NOT solve the problem.)