Singular Sturm-Liouville, Fredholm’s Alternative, and Integral Equations

0. Reading: Haberman, sections 5.5 (Sturm-Liouville), 9.4.2 (Fredholm alternative). For integral equations, see references listed in the Lecture 10 handout.

1. Legendre’s equation: Consider \( \text{bounded} \) solutions of the Sturm-Liouville eigenvalue problem on the interval \(-1 \leq x \leq 1\) for

\[
\frac{d}{dx} \left[(1 - x^2) \frac{d\phi}{dx}\right] + \lambda \phi = 0.
\]

(a) What are \( p(x), q(x), \sigma(x) \)? Explain why no boundary conditions need to be specified.

(b) Viewing (1) as \( L\phi + \lambda\sigma\phi = 0 \), use the inner product relation for the adjoint \( \langle v, Lu \rangle \) to directly compute \( L^* \) by integration by parts and show that the boundary terms will always vanish for any smooth, bounded functions \( u,v \).

(c) What orthogonality relation do the eigenfunctions satisfy?

(d) Let \( \phi_0(x) \equiv 1 \). This is the eigenfunction for \( \lambda_0 = 0 \), \( L\phi_0 = -\lambda_0\sigma\phi_0 \). Let \( w_k(x) \equiv x^k \) for \( k = 1, 2, 3, \ldots \). Use the Gram-Schmidt orthogonalization process\(^1\) with the orthogonality relation from (c) to construct the eigenfunctions \( \phi_1, \phi_2, \phi_3 \) from \( w_1, w_2, w_3 \).

(e) Evaluate \( L\phi_k \) for \( k = 1, 2, 3 \) for your \( \phi_k \) from (d) to determine the corresponding eigenvalues \( \lambda_k \).

(f) Verify by direct substitution that \( u(x) = A\ln \left( \frac{1+x}{1-x} \right) + B \) is a solution of \( Lu = 0 \) for any values of \( A \) and \( B \). Is the log term bounded on the whole domain?\(^3\) What value for \( A \) makes \( u(x) \) a bounded solution for this problem?

(g) Find the general solution of \( Lu = 1 \) by direct integration of this ODE.

Is there an “acceptable” (bounded) solution for this problem?

(h) In order for the inhomogeneous problem \( Lu = f(x) \) to have a solution in the form of an eigenfunction expansion, \( u(x) = \sum_k c_k\phi_k(x) \), what solvability condition must \( f(x) \) satisfy?

2. Use the Fredholm alternative theorem and the solvability condition to determine the parameter values (\( A \) or \( A,B \)) that yield existence of a solution for each inhomogeneous boundary value problem: (Show all work! but do NOT calculate the complete solution \( u(x) \).)

(a) For \( 0 \leq x \leq \pi \):

\[
\frac{d^2u}{dx^2} + 8\frac{du}{dx} + 16u = 8xe^{-4x}. \quad u'(0) + 4u(0) = 5A + 3 \quad u'(\pi) + 4u(\pi) = A.
\]

Hint: Recall HW3, Q3(b), \( L^*v = v'' - 8v' + 16v \). What is the adjoint eigenfunction for \( \lambda = 0? \)

(b) For \( 0 \leq x \leq 2\pi \):

\[
\frac{d^2u}{dx^2} + u = A\sin(x/4) + B\sin(x/2) + 4x^2 \quad u(0) = u(2\pi) \quad u'(0) = u'(2\pi).
\]

Hint: The linear operator is self-adjoint and has two (separate) linearly independent adjoint solutions for \( \lambda = 0 \).

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\(^1\)Recall from linear algebra: To construct an orthogonal set of vectors \( \{\vec{v}_k\} \) from an ordered set of linearly independent vectors \( \{\vec{u}_k\} \), subtract-off from each \( \vec{u}_k \) all of the projections of \( \vec{u}_k \) onto the previously generated \( \vec{v}_j \) \((j = 0, \ldots, k - 1)\) vectors:

\[
\vec{v}_0 = \vec{u}_0 \\
\vec{v}_1 = \vec{u}_1 - \frac{\langle \vec{u}_1, \vec{v}_0 \rangle}{\langle \vec{v}_0, \vec{v}_0 \rangle} \vec{v}_0 \\
\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_0 \rangle}{\langle \vec{v}_0, \vec{v}_0 \rangle} \vec{v}_0 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 \\
\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_0 \rangle}{\langle \vec{v}_0, \vec{v}_0 \rangle} \vec{v}_0 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 \\
\]

and so on. Observe that this will yield \( \langle \vec{v}_k, \vec{v}_j \rangle = 0 \) for any \( k \neq j \).

\(^2\)The eigenfunctions are called the Legendre polynomials, \( P_k(x) \) (also called the Legendre functions of first kind).

\(^3\)This term is also written as \( AQ_0(x) \) where \( Q_0(x) \) is called the zero-th Legendre function of second kind, part of \( Q_k(x) \) family.
3. Solution of FIE’s (v1.0): Solutions of first kind Fredholm integral equations can be obtained via the method of undetermined coefficients by direct substitution of \( u(x) = \sum_{j=1}^{n} d_j \alpha_j(x) \) into the equation. Find a solution or show that no solution exists for

(a) \( \int_0^1 (x - 5x^2t^3)u(t) \, dt = x - 4x^2 \)

(b) \( \int_0^1 \sin(2\pi x - \pi t)u(t) \, dt = 3 \cos(\pi x) - \sin(2\pi x) \)

(c) \( \int_0^\infty \left[ e^{-5x-t} - 6e^{-3x-4t} \right] u(t) \, dt = 4e^{-3x} \)

(d) \( \int_0^\pi \left[ 12t^2 \cos^2(x) - 8t \sin^2(x) \right] u(t) \, dt = 1 \)

4. Solution of FIE’s (v2.0)

Re-examine the integral operator \( Lu \equiv \int_0^1 (x - 5x^2t^3)u(t) \, dt \) (from Question 3a):

(a) Find the eigenvalues of finite multiplicity and their eigenfunctions for \( L\phi_j = \lambda_j \phi_j \).

(b) Write the adjoint operator and determine the adjoint eigenfunctions for the eigenvalues of finite multiplicity for \( L^* \psi_j = \lambda_j \psi_j \).

(c) For \( f(x) = x - 4x^2 \), determine \( c_1, c_2 \) in the expansion \( u(x) = c_1 \phi_1 + c_2 \phi_2 \) for the solution of \( Lu = f \) using \( c_j = \frac{\langle \psi_j, f \rangle}{\lambda_j \langle \psi_j, \phi_j \rangle} \) to show that it matches your solution from Question 3(a).

(d) Fredholm integral operators of first kind have zero as an eigenvalue of infinite multiplicity, \( \lambda_\infty = 0 \). Demonstrate this by constructing a set of adjoint eigenfunctions satisfying \( L^* \psi_\infty^m = 0 \) using the form \( \psi_\infty^m(x) = 1 + b_1x + b_2x^m \) for \( m = 2, 3, \cdots \). Find \( b_1(m), b_2(m) \).

5. Solution of FIE’s (concluded)

Consider the Fredholm integral operator of second kind \( Lu \equiv 4u(x) + \int_0^1 \sin(2\pi x - \pi t)u(t) \, dt \):

(a) Find the eigenvalues of finite multiplicity and their eigenfunctions for \( L\phi_j = \lambda_j \phi_j \).

(b) Show that \( \lambda_\infty = 4 \) is of infinite multiplicity by showing that there is an infinite set of sine’s and cosine’s that satisfy \( L\phi = 4\phi \).

(c) Find the unique solution of \( Lu = 4 + 8x \) by direct substitution of \( u(x) = p + qx + \sum_{j=1}^{n} d_j \alpha_j(x) \)

into \( Lu \) to determine unknown coefficients \( p, q, d_j \).

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Math 551 Test 1: [Week of Sept 23, 2020] Covers Lectures 2–9, Homeworks 2–4:


Tests from previous years will be posted to help guide your studying.

No Homework will be due the week of the test to allow for your time to study and take the test.

The test will be a timed take-home test in Gradescope out of lecture times. Details will follow.

No books, no calculators. But you can use the ‘basic mathematics summary’ and one letter-sized sheet (2 sides) of your own handwritten notes (to be scanned in with the rest of your work).