Math 551: Applied PDE and Complex Variables Fall 2023

Problem Set 3 **Assigned Fri Sep 15** **Due Fri Sep 22**

Eigenvalue and Boundary Value Problems for Linear ODE's

0. Reading: Haberman, Sections 5.3, 5.5. Lecture notes 5, 6 and the review sheet on LCC/CE eqns.

1. Sturm-Liouville form for all* second-order equations: Consider the basic eigenvalue problem $\mathrm{L}\phi = -\lambda\phi$ for general second-order linear operators,

$$\mathrm{L}u \equiv A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u \qquad \Longrightarrow \qquad A(x)\frac{d^2\phi}{dx^2} + B(x)\frac{d\phi}{dx} + C(x)\phi = -\lambda\phi \tag{1}$$

where $A(x), B(x), C(x)$ are given functions.[1]
Equation (1) can always be put into weighted Sturm-Liouville form, $\widetilde{\mathrm{L}}\phi = -\lambda\sigma\phi$, with the same $\{\lambda, \phi\}$,

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi = -\lambda\sigma(x)\phi, \tag{2}$$

even if the original L is not formally self-adjoint in the standard L^2 inner product.

(a) Show this by determining $p(x), q(x), \sigma(x)$ in terms of $A(x), B(x), C(x)$.
Hint: Expand the product rule term in (2), multiply the equation across by an unknown integrating factor function, $m(x)$, and finally match that equation term by term with (1) to obtain four equations for $p(x), q(x), \sigma(x), m(x)$. Find $p(x)$ first.

(b) If you are given the eigenfunctions $\phi_k(x)$ for the Sturm-Liouville operator $\widetilde{\mathrm{L}}$, how can you write the ψ_k adjoint eigenfunctions for L using ϕ_k? Hint: What is L^*v in terms of $\widetilde{\mathrm{L}}$?

(c) For Sturm-Liouville problems, $\widetilde{\mathrm{L}}u = f(x)$ on $a \le x \le b$ with inhomogeneous boundary conditions, derive **Green's formula** (see Haberman eqn 5.5.8), namely evaluate $\langle \phi_k, \widetilde{\mathrm{L}}u\rangle_2 = \widetilde{B}_k + \langle \widetilde{\mathrm{L}}\phi_k, u\rangle_2$.
(Yes, just IBP^2 like HW2Q4e, but this comes up A LOT, so keep this result around and re-use it when needed:)

2. Consider the eigenvalue problem on $0 \le x \le \pi$,

$$\frac{d^2\phi}{dx^2} + 6\frac{d\phi}{dx} + (9+\lambda)\phi = 0,$$
$$\phi'(0) + 3\phi(0) = 0, \qquad \phi'(\pi) + 3\phi(\pi) = 0.$$

(a) Assume λ to be a non-negative constant. What is the general solution of the homogeneous ODE. Consider the cases $\lambda > 0$ and $\lambda = 0$ separately.

(b) Apply the boundary conditions to determine the eigenvalues and eigenfunctions.

(c) What is the adjoint problem? Obtain the adjoint eigenfunctions.

(d) What are the functions p, q, σ that put this problem in standard Sturm-Liouville form? (see Prob 1)

(e) Using (a) and (b) write the simplest forms for the integrals for the standard inner product $\langle \phi_k, \psi_j\rangle_2$ and the weighted inner product $\langle \phi_k, \phi_j\rangle_\sigma$ orthogonality conditions and show that they are equivalent.

(f) Obtain the coefficients c_k in the expansion of the function $u(x) = \sum_{k=0}^{\infty} c_k\phi_k(x)$ that solves the BVP:

$$\frac{d^2u}{dx^2} + 6\frac{du}{dx} + 4u = \frac{1}{4}\,e^x,$$
$$u'(0) + 3u(0) = 13 \qquad u'(\pi) + 3u(\pi) = 7.$$

Hint: Be careful about c_0 vs $c_{1,2,3\dots}$
Hint: If $\phi(x)$ is a solution of $\mathrm{L}\phi = -\lambda\phi$ then it is also a solution of $\mathrm{L}\phi - c\phi = -\widehat{\lambda}\phi$ where $\widehat{\lambda} = \lambda + c$.
Shifting the eigenvalue keeps the same eigfcn! *(continued)*

[1] Haberman's problem 5.3.3 is similar. This is also the ODE analogue of the weighted inner product for the non-symmetric 2×2 matrix, problem 5.5A.3, Homework# 1 Problem 2.

3. Another inhomogeneous boundary value problem:

   (a) Find the general homogeneous solution of the Cauchy-Euler equation

   $$x^2\frac{d^2\phi}{dx^2}+7x\frac{d\phi}{dx}+(9+\alpha)\phi=0,$$

   where α is assumed to be a positive constant.

   (b) Use (a) to determine the eigenvalues and eigenfunctions of the Sturm-Liouville problem on $1\le x\le e$

   $$\frac{d}{dx}\left(x^7\frac{d\phi}{dx}\right)+\lambda x^5\phi=0 \qquad \phi(1)=0 \qquad \phi(e)=0.$$

   (c) Use (b) to obtain the eigenfunction expansion for the solution of the inhomogeneous problem

   $$\frac{d}{dx}\left(x^7\frac{du}{dx}\right)=x^2 \qquad u(1)=0 \qquad u(e)=4.$$

4. Reducing Gibbs' phenomenon: It can be useful to obtain the solution of a BVP in a form that minimizes Gibbs' phenomenon issues at the boundaries. This question illustrates how to do that by breaking up the solution into two pieces: an eigenfunction expansion for the forced solution and a separate particular solution (not an eigen-expansion) that will satisfy the inhomogeneous boundary conditions.
   The problem

   $$\mathrm{L}u(x)=f(x) \qquad BC_1u(a)=c \qquad BC_2u(b)=d \tag{3}$$

   can be solved in two steps by writing the solution as $\boxed{u(x)=u_B(x)+u_F(x)}$ where

   - The "boundary solution": $u_B(x)$ satisfies the equation with a modified forcing function and with the original inhomogeneous boundary conditions

     $$\mathrm{L}u_B(x)=m(x) \qquad BC_1u_B(a)=c \qquad BC_2u_B(b)=d \tag{4}$$

     $m(x)$ can be chosen to yield a simple form for $u_B(x)$.
   - The "forced solution": $u_F(x)$ satisfies the equation including the original forcing, but with homogeneous boundary conditions

     $$\mathrm{L}u_F(x)=f(x)-m(x) \qquad BC_1u_F(a)=0 \qquad BC_2u_F(b)=0 \tag{5}$$

     Note that adding together equations (4) and (5) yields (3) for the solution $u(x)$.

   Apply this to the example from Lecture 5: $\boxed{\frac{d^2u}{dx^2}=9e^{4x}, \qquad u(0)=-5, \qquad u(1)=-7}$

   For this example, use the choice $m(x)\equiv 0$:

   (a) Find the polynomial function $u_B(x)$ that solves $u_B''=0, \quad u_B(0)=-5, \quad u_B(1)=-7.$
   Hint: Do not use eigenfunction expansions, just simple integration.

   (b) Find $u_F(x)$ as an eigenfunction expansion that solves $u_F''=9e^{4x}, \quad u_F(0)=0, \quad u_F(1)=0.$

   (c) Show that your overall solution $u=u_B+u_F$ matches the Fourier coefficients c_k, equation (21), found for $u(x)$ in Lecture 5. (Find the Fourier series of $u_B(x)$.)

   (d) The series for $u_F(x)$ still has some Gibbs' phenomena in it, but at a "weaker level". Based on its series coefficients for $k\to\infty$, how smooth is $u_F(x)$? (What is the lowest order derivative of $u_F(x)$ that is discontinuous anywhere?)