Linear Algebra Review


Please SHOW ALL WORK leading up to your solutions – intermediate steps are important (and will get you partial credit)!

Unexcused late homeworks will not be accepted. Any extensions or excuses must be requested before the due date.

Office hours: We will vote (doodle poll) on regular office hours for times on Mondays and Tuesdays. You can always email me at any time with your questions or to request to schedule a time to zoom-meet with me.

0. Reading: Haberman, Appendix to section 5.5 (pp. 178–183).

1. The eigen-expansion method for solving systems of linear equations

Consider the matrix equation $Lu = b$:

$$
\begin{pmatrix} -6 & 2 & 7 \\ 8 & 0 & -7 \\ 4 & 4 & 6 \end{pmatrix} u = \begin{pmatrix} -34 \\ 22 \\ -36 \end{pmatrix}.
$$

You are given that the eigenvalues for $L\phi = \lambda \phi$ are $\{\lambda_1 = -8, \lambda_2 = 6, \lambda_3 = 2\}$.

(a) Find the eigenvectors $\{\phi_1, \phi_2, \phi_3\}$, scale them so that the first component of each is equal to one.

(b) Show that $\{\phi_1, \phi_2, \phi_3\}$ are not orthogonal, but they are linearly independent.

Hint: To test for orthogonality, calculate the dot products $\phi_i \cdot \phi_j$.

For linear independence, recall the definition: vectors are linearly independent if $c_1\phi_1 + c_2\phi_2 + c_3\phi_3 = 0$ only for $c_1 = c_2 = c_3 = 0$. This can be written as a matrix-vector equation, $\Phi c = 0$, with the vectors being the columns of $\Phi$. What does $\det(\Phi)$ tell you about the uniqueness of the solution?

(c) Find the adjoint eigenvectors $\{\psi_1, \psi_2, \psi_3\}$. Scale them so that the first component of each is equal to one.

(d) Determine the expansion coefficients $c_k$ and compute the solution $u = \sum_k c_k \phi_k$ to confirm that this agrees with $u = L^{-1}b = (1, -7, -2)^T$.

2. Linear algebra with a different inner product\(^1\)

(a) Haberman page 183, Problem 5.5A.3.

(b) Find the adjoint eigenvectors of $A$ with respect to the regular dot product.

Verify orthogonality by calculating the inner products $\phi_i \cdot \phi_j$.

(c) Let $M = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$. Show that the book’s “$a \cdot b$” dot product can be written as a “weighted inner product” defined by $\langle a, b \rangle \equiv a \cdot M b$ (or $= a_1 b_1 \sigma_1 + a_2 b_2 \sigma_2$ with $\sigma_1 = m_{1,1}$ and $\sigma_2 = m_{2,2}$).

(d) Find the positive diagonal matrix $C$ such that $M = C^2$.

(e) Multiply the eigenvalue equation $A\phi = \lambda \phi$ on the left by $M$ to get $MA\phi = \lambda M\phi$. Write $M = C^2$ and $\phi = C^{-1}y$ in this equation and re-arrange to give an eigenvalue/eigenvector equation for $y$, $By = \lambda y$, where $B$ is a real symmetric matrix, $B = B^T$. What is the matrix $B$? (expressed in terms of $A$, $C$ and also specific numbers) Show that the vectors $y$ are eigenvectors for $By = \lambda y$, where $B$ is a symmetric matrix – Find $B$.

There will be many problems coming soon which will use weighted inner products.
(f) Explain why the orthogonality relation for the $y$'s is $y_1 \cdot y_2 = 0$ and use this to justify the orthogonality of the $\phi$'s in the weighted inner product from part (c).

Hint: Use $y = C\phi$ in the dot product.

3. Linear algebra with complex-valued vectors

For vectors whose entries are complex numbers ($x, y \in \mathbb{C}^n$) the inner product is defined as $\langle x, y \rangle \equiv x^T \bar{y}$, where $\bar{y}$ is the complex conjugate of vector $y$, conjugated entry by entry in the vector.

The conjugate of a complex number, $z = a + ib$ is defined as $\bar{z} = a - ib$ where $i^2 = -1$. Note that $z + w = \bar{z} + \bar{w}$ and $\bar{zw} = \bar{z} \bar{w}$ for all complex numbers $z, w$.

(a) Show that the “real inner product” ($x^T y$) does not satisfy the norm property for complex vectors.

Hint: What is the value of $x^T x$ for the vector $x = (1, i)^T$?

(b) Let $x = (a + ib, c + id)^T$, where $a, b, c, d$ are real numbers. Show that the “complex inner product” is a norm, with $|x|^2 = \langle x, x \rangle \geq 0$.

(c) How is the value of the complex inner product $\langle x, y \rangle$ related to the value of $\langle y, x \rangle$?

(d) If $A$ is a matrix with complex-valued entries, what is the formula for the adjoint $A^*$ satisfying $\langle Ax, y \rangle = \langle x, A^* y \rangle$ with respect to the complex inner product? How are the adjoint eigenvalues $\gamma_k$ related to the $\lambda_k$ of $A$?

(e) Find $\{\lambda_k, \phi_k\}$ and $\{\gamma_k, \psi_k\}$ for $A = \begin{pmatrix} i & -1 \\ 2 & i + 2 \end{pmatrix}$ and show that $\phi_1 \perp \psi_2$ and $\phi_2 \perp \psi_1$.

(f) Haberman page 183, Problem 5.5A.6.

Hint: For part (a) of this problem, consider the complex inner product of the matrix times an eigenvector against the same eigenvector, and consider what happens when you factor a constant out of the inner product (from the first vs. second factors) to end up showing that $(\lambda_k - \overline{\lambda_k}) = 0$. (This works for both complex-Hermitian matrices and real-symmetric matrices.)

4. Solution of initial value problems for matrix-vector ODE systems

Haberman page 183, Problem 5.5A.4, part (a).

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2Complex inner products will be used in Fourier transforms.