Inviscid, Incompressible, Irrotational flows

0. Reading: Acheson, sections 1.3, 1.5, 4.1, 4.2, 4.3.

1. Consider each two-dimensional steady flow:
   (a) \( u(x, y) = 2x - 6x^2 + y, \quad v(x, y) = x + 3y^2 \). Determine the velocity potential. Show that a streamfunction does not exist.
   (b) \( u(x, y) = x^2 - 9y^2 + 5, \quad v(x, y) = -2xy - 2x \). Determine the streamfunction. Show that a velocity potential does not exist.
   (c) Determine \( u(x, y) \) and the complex potential \( w(z) \) for an incompressible, irrotational flow with \( v(x, y) = 4x - 6y - 6xy - 8 \).

2. Consider the complex potential \( w(z) = \sqrt{z} \), describing the flow around a half-line (the positive \( x \)-axis). Show that the streamlines are parabolas.

3. Consider the 2D unsteady flow for a moving irrotational vortex,
   \[
   \mathbf{u}(x, y, t) = \begin{pmatrix} -y \frac{x - at}{(x - at)^2 + y^2} & \frac{y}{(x - at)^2 + y^2} \end{pmatrix}
   \]
   (a) Determine the pressure \( p(x, y, t) \).
   (b) Show that this flow satisfies the 2D unsteady Euler equations (use of computer algebra is encouraged, provide complete printouts).

4. Show that any (non-degenerate) stagnation point of a steady two-dimensional incompressible irrotational flow,
   \[
   \frac{dx}{dt} = u(x, y), \quad \frac{dy}{dt} = v(x, y),
   \]
   must be a saddle point, with \( \lambda_+ = -\lambda_- \) in its linear stability analysis.

5. For two-dimensional incompressible flows with streamfunction \( \psi(x, y, t) \):
   (a) Show that the velocity field automatically satisfies the incompressibility condition, for any \( \psi \).
   (b) Write down the scalar vorticity in terms of the streamfunction.
   (c) Show that the 2D inviscid scalar vorticity equation can be written as
       \[
       \frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} = 0
       \]
   (d) Show that all steady solutions from (c) satisfy \( \nabla^2 \psi = q(\psi) \) for some function \( q \).
      i. What choice of \( q \) corresponds to irrotational flow?
      ii. Solve \( \nabla^2 \psi = -\psi \) for axisymmetric solutions, \( \psi = \psi(r) \).
      iii. Solve \( \nabla^2 \psi = e^\psi \) in one-dimension, \( \psi = \psi(x) \).
   Hints: \( ab - cd = \det(A) \) and the determinant of a \( 2 \times 2 \) matrix is zero if the rows are proportional to each other.
   (e) Extend your result from (c) to the viscous case to write the “vorticity-streamfunction” form of the 2-D Navier-Stokes equations.
   Describe how you might apply no-slip and no-flux boundary conditions if you were to use only \( \omega, \psi \).