Flow Kinematics

*kinematics*: the science of pure motion, considered without reference to the matter or objects moved or the force producing or changing the motion. [i.e. $\vec{u}$ is given] – OED

0. Reading: Acheson, sections 1.2, 6.4 (pp. 212–214), also watch the NCFMF film on *Flow Visualization*.


2. Consider the two-dimensional velocity field,

$$u(x, y, t) = \frac{x}{1 + t}, \quad v(x, y, t) = -y \cos(t).$$

Obtain the analytic expressions for $y = y(x)$ and plot the curves together on a single $xy$ graph:

(a) The pathline for a particle starting from $(1, 1)$ at $t = 0$.

(b) The instantaneous streamline at $t = 0$ for a particle starting from $(1, 1)$.

(c) The streakline at $t = 0$ for all particles that would have passed through the point $(1, 1)$.

3. Consider the phase plane system of linear ordinary differential equations,

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy,$$

where $a, b, c, d$ are real-valued constants. This system can also be written in vector form as

$$\frac{d\vec{x}}{dt} = D\vec{x}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Interpret this ODE system as describing a two-dimensional fluid flow, $d\vec{x}/dt = \vec{u}$ with $\vec{u}(\vec{x}) = D\vec{x}$.

(a) Decompose the flow into area dilations, strains, and rotations.

(b) What is the rate of change of area for this flow?

(c) What is the vorticity for this flow?

(d) Consider the two linear flow fields,

$$D_1 = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} \frac{11}{5} & \frac{8}{5} \\ \frac{8}{5} & -\frac{1}{5} \end{pmatrix}$$

i. Show that the $D$’s share the same eigenvalues.

ii. Write down the solutions $x(t), y(t)$ for each flow starting from initial condition $\vec{x}(t = 0) = (2, 1)$.

iii. Use (a) to describe the qualitative differences between the two flows.

iv. Use (ii) to determine the asymptote line, $y = kx$, that is approached for $t \to \infty$.

v. Determine the eigenvectors of the rate-of-strain tensor ($E$) for each flow. These eigenvectors point in directions called the *principal axes* of the flow. Show the asymptotes in (iv) are principal axes in directions of stretching.

vi. The two following flows also share the same eigenvalues,

$$D_3 = \begin{pmatrix} 2 & -1 \\ 8 & -2 \end{pmatrix}, \quad D_4 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

How do the $x(t), y(t)$ pathlines differ in these two flows?
4. Consider the two-dimensional steady flow described by the autonomous phase plane system:

\[
\begin{align*}
\frac{dx}{dt} &= 1 - \frac{x^2 - y^2}{(x^2 + y^2)^2}, \\
\frac{dy}{dt} &= -\frac{2xy}{(x^2 + y^2)^2}
\end{align*}
\]

(a) Show that the flow is incompressible and irrotational.

(b) **Stagnation points** (also called equilibrium points or fixed points) are positions where the flow velocity is zero. Find the stagnation points of the flow and classify the linearized behaviors in their neighborhoods (i.e. the eigenvalues of \(D\) and whether the equilibrium point is a node, center, saddle, or spiral).

(c) Show that points starting on the unit circle, \(x^2 + y^2 = 1\), remain on the circle for all times.

(d) What information does calculating \(\nabla \cdot (y^2 \hat{u})\) give you about the existence of periodic solutions in the flow according to the **Bendixson-Dulac theorem**? (look it up)

5. Consider the velocity field,

\[
u(x, y, z, t) = yz + t, \quad v(x, y, z, t) = xz + t, \quad w(x, y, z, t) = xy.
\]

Write the velocity gradient tensor. Show that this velocity field is incompressible and irrotational for all times.

6. Derive the general expression for the local instantaneous area strain rate in a 3D flow \(\vec{u} = \vec{u}(\vec{x}, t)\) using the following steps:

Consider three points moving with the flow, \(\vec{x}_1(t), \vec{x}_2(t),\) and \(\vec{x}_3(t)\). Let these points be adjacent corners of a parallelogram with edge vectors, \(\vec{p} = \vec{x}_1 - \vec{x}_2\) and \(\vec{q} = \vec{x}_3 - \vec{x}_2\). Recall that the area of a parallelogram with edges \(\vec{p}, \vec{q}\) is given by the length of the cross product vector, \(\text{Area} = A(t) = |\vec{p} \times \vec{q}| = |\vec{p}| |\vec{q}| \sin \theta\), where \(\theta\) is the angle between the edge vectors.

(a) Evaluate \(\frac{1}{A} \frac{dA}{dt}\). Your answer will be in terms of \(\vec{p}, \vec{q}\), \(A\) and \(\vec{u}\) at \(\vec{x}_{1,2,3}\).

Hint: some miscellaneous vector algebra/calculus,

\[
|\vec{a}| = (\hat{a} \cdot \hat{a})^{1/2} \quad \vec{a} = |\vec{a}| \hat{a} \quad \hat{a} = \vec{a} / |\vec{a}| \quad \text{(unit direction vector)}
\]

\[
\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \quad (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})
\]

\[
\frac{d}{dt}(\vec{a} \cdot \vec{b}) = \frac{d\vec{a}}{dt} \cdot \vec{b} + \vec{a} \cdot \frac{d\vec{b}}{dt} \quad \text{(and other versions of the product rule)}
\]

\[
\frac{d}{dt} f(\vec{a}(t)) = \nabla f(\vec{a}) \cdot \frac{d\vec{a}}{dt} \quad \text{(and other versions of the chain rule)}
\]

(b) Write the lengths of the edges as \(h = |\vec{p}|\) and \(k = |\vec{q}|\). The edge vectors can then be written in terms of unit direction vectors as \(\vec{p} = h\hat{p}, \vec{q} = k\hat{q}\). Take the limit of (a) as \((h,k) \to 0\) with \(\vec{p}\) and \(\vec{q}\) held fixed. Simplify the answer as much as possible. Your answer will be in terms of \(\hat{p}, \hat{q}\), and \(\nabla \vec{u}(\vec{x}_2)\). Hint:

\[
D_h f(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\hat{a}) - f(\vec{x}_0)}{h} \quad \text{(the limit definition of the derivative)}
\]

\[
D_h f(\vec{x}) = \nabla f \cdot \frac{\hat{a}}{|\hat{a}|} \quad \text{(the directional derivative)}
\]

(c) Show that for a 2D flow, \(\vec{u} = (u(x, y, t), v(x, y, t), 0)\), with \(\vec{x}_1, \vec{x}_2, \vec{x}_3\) in the \(xy\) plane, part (b) yields the divergence \(\nabla \cdot \vec{u} = u_x + v_y\).\footnote{Given how complicated this “2D” problem is, you can imagine that this approach to calculating the 3D problem for the volumetric strain rate of a parallelepiped, Volume = \(V(t) = |\vec{r} \cdot (\vec{p} \times \vec{q})|\), will be MUCH more laborious (DO NOT TRY IT).}