1. (23 pts) Evaluate the following complex integral on the closed counterclockwise-oriented contour $C$ given by the circle of radius $\sqrt{2}$ whose center is $z_c = \frac{1}{2} + i$:

$$\oint_C \frac{e^{3z}}{(z - 1)^2(z^2 + 1)\sin(\frac{1}{2}\pi z)}
dz$$

Note that $\sin(z)$ has zeros only on the real axis.

(a) (5 pts) Identify the positions and types of the singularities inside $C$.

(b) (18 pts) Calculate the residues and evaluate the integral. Simplify your answer as much as possible.

2. (26 pts) Use complex contour integration to calculate the value of the integrals:

$$I_1 = \int^{\infty}_{-\infty} \frac{dx}{(x - 1)^2 + 1} \quad I_2 = \int^{\infty}_{0} \frac{dx}{(x - 1)^2 + 1}$$

Show all work and justify your choice of contour for each. ($I_1$: 7 pts, $I_2$: 19 pts)

3. (34 pts) Calculate the value of the real integral

$$I = \int^{\infty}_{-\infty} \frac{x^2e^{-x/2}}{1 + e^{-x}}
dx \quad \text{using the contour integral} \quad \oint_C \frac{z^2e^{-z/2}}{1 + e^{-z}}
dz$$

where $C$ is a box contour with top segment given by $z = x + ib$ for some positive constant $b$. Note that $\int^{\infty}_{-\infty} \frac{e^{-x/2}}{1 + e^{-x}}
dx = \pi$

(a) (5 pts) Sketch the contour and parametrize all segments of the contour integral.

(b) (4 pts) Determine a value for $b$ where the integral on the top segment can be related to $I$.

(c) (8 pts) Show that the contribution of the integral on each vertical segment vanishes.

(d) (17 pts) Evaluate the contour integral and give the value for $I$ simplified as much as possible.

4. (17 pts) Evaluate the trigonometric integral using a complex contour integral:

$$I = \int^{2\pi}_{0} \frac{e^{\cos\theta + i\sin\theta}}{5 - 4\cos\theta}
d\theta$$

(a) (5 pts) Write the contour integral, sketch the contour, and identify the singularities inside.

(b) (12 pts) Evaluate the contour integral and give the value for $I$. Simplify your answer as much as possible.