Math 211, Fall 2009, Test 3

1. (26 pts) Evaluate the following complex integral on the closed counter-clockwise contour $C$ given by the circle $|z - 1| = 3$: 

$$\oint_C \frac{1}{z^2 \sin z} \, dz$$

Note that $|\sin z|$ does not vanish if $\text{Im}(z) \neq 0$.

(a) (6 pts) Identify the positions and types of the two singularities inside $C$.

(b) (20 pts) Calculate the residues and evaluate the integral. Simplify your answer as much as possible.

2. (24 pts) Evaluate the trigonometric integral using a complex contour integral:

$$I = \int_0^\pi \frac{1}{4 \sin^2 \theta + 12 \cos^2 \theta} \, d\theta$$

(a) (12 pts) Write the contour integral, sketch the contour, and identify the singularities inside.

(b) (12 pts) Evaluate the contour integral and give the value for $I$. Simplify your answer as much as possible.

3. (20 pts) Use a complex contour integral to calculate the value of the integral

$$I = \int_0^\infty \frac{x \sin x}{x^2 + a^2} \, dx$$

where $a > 0$.

(a) (6 pts) Write the contour integral and sketch the contour.

(b) (6 pts) Justify your choice of closed contour.

(c) (8 pts) Evaluate the contour integral and give the value for $I$. Simplify your answer as much as possible.

4. (30 pts) Calculate the value of

$$I = \int_0^\infty \frac{\sqrt{x}}{1 + x^4} \, dx$$

using the contour integral

$$\oint_C \frac{\sqrt{z}}{1 + z^4} \, dz$$

where the branch cut for the square root is taken along the positive real axis and $C$ is a sector contour that is indented at the origin.

(a) (8 pts) Determine an appropriate closed contour for this problem and sketch it.

(b) (5 pts) Show that the portion of the contour near the origin does not contribute to the value of the integral.

(c) (5 pts) Show that the portion of the contour at large $|z|$ does not contribute to the value of the integral.

(d) (12 pts) Evaluate the contour integral and give the value for $I$ simplified as much as possible.

Have a good Thanksgiving break!