Math 211, Fall 2009, Test 1

1. (50 pts) You are given that for any value of the constant $\alpha$, the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\alpha x^2 - \frac{1}{4}) y = 0 \quad \text{has general solution} \quad y(x) = c_1 \frac{\sin(\sqrt{\alpha} x)}{\sqrt{x}} + c_2 \frac{\cos(\sqrt{\alpha} x)}{\sqrt{x}}.
$$

Consider bounded solutions of the eigenvalue problem for $\phi(x)$ on $0 \leq x \leq 1$:

$$x^2 \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} - \frac{1}{4} \phi = -\lambda x^2 \phi, \quad \phi(1) = 0. \tag{1}
$$

(a) (7 pts) Determine the eigenvalues and eigenfunctions.

(b) (8 pts) Determine the adjoint operator and adjoint boundary conditions.

(c) (3 pts) Determine $p(x), q(x), \sigma(x)$ that put the problem into the form of a singular Sturm-Liouville equation, $(p(x)\phi')' + q(x)\phi = -\lambda \sigma(x)\phi$.

Hint: Divide the equation across by $x$.

(d) (3 pts) Use (a,c) to write the integral in simplest form for the orthogonality relation between $\phi_j$ and $\phi_k$ with respect to $\sigma(x)$ for $j \neq k$. (DO NOT evaluate the integral.)

(e) (5 pts) For a function $f(x)$ given on $0 \leq x \leq 1$, write the formula for the coefficients in its expansion, $f(x) = \sum_k f_k \phi_k(x)$, in terms of integrals in simplest form with the eigenfunctions from (a).

(f) (24 pts) Find the coefficients in the eigenfunction expansion of the solution, $u(x) = \sum_k c_k \phi_k(x)$, of the boundary value problem

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} - \frac{1}{4} u = x^{5/2}, \quad u(1) = 1.
$$

Simplify your answer as much as possible.

Hint: First, divide the problem across by $x$ to make the LHS match Sturm-Liouville form, and then solve that equation.

2. (50 pts) Consider the linear operator

$$Lu \equiv \int_0^1 \left[16x - 30x^2t^2\right] u(t) \, dt. \tag{2}
$$

(a) (17 pts) Determine all of the eigenvalues and determine the eigenfunctions for the eigenvalues of finite multiplicity in $L\phi = \lambda \phi$.

(b) (12 pts) Write the adjoint operator. Determine the adjoint eigenfunctions for the eigenvalues of finite multiplicity.

(c) (21 pts) Let $\beta$ be a real constant. Consider the second kind Fredholm integral equation for $u(x)$,

$$Lu + \beta u(x) = 1 - 8x^3.
$$

i. (3 pts) For what value(s) of $\beta$ does this equation have no solution?

ii. (3 pts) For what value(s) of $\beta$ does this equation have only one solution?

iii. (3 pts) For what value(s) of $\beta$ does this equation have infinity many solutions?

iv. (12 pts) Find a solution for $\beta = 1$.

Note: Part (iv) can be done independently of parts (i–iii).