Problem Set 5  Assigned Mon Sep 28  Due Weds Oct 12

Green’s Functions for ODE BVP


Optional review session: ???.

0. Read Haberman, section 9.3 (pages 385–399). (My s is the same as Haberman’s x0.)

1. Haberman, page 400. Problem 9.3.6(a) and sketch g(x, s) vs. x for some fixed value of s.

2. (Verifying Mercer’s Theorem) Consider the inhomogeneous boundary value problem on \(0 \leq x \leq \pi/2\),

\[
\frac{d^2 u}{dx^2} + u = f(x) \quad u(0) = 0 \quad u(\pi/2) = 0.
\]

(a) Work-out the piecewise-defined Green’s function used in writing the solution as

\[
u(x) = \int_0^\pi/2 g(x, s)f(s) \, ds.
\]

(b) Find the eigenfunctions and eigenvalues for

\[
\frac{d^2 \phi}{dx^2} + \phi = -\lambda \phi, \quad \phi(0) = 0, \quad \phi(\pi/2) = 0.
\]

(c) Calculate the coefficients in the eigenfunction expansion of \(g(x, s)\) [the piecewise-defined function used in (2)],

\[
g(x, s) = \sum_{k=1}^{\infty} g_k \phi_k(x)
\]

Hint: Think of \(g\) just as a function of \(x\) (with \(s\) as a fixed parameter).

(d) Write the solution of problem (1) in terms of an expansion using the eigenfunctions,

\[
u(x) = \sum_{k=1}^{\infty} c_k \phi_k(x),
\]

and note the relation between \(g_k\) and \(c_k\).


4. Find the piecewise-defined Green’s function for the problem

\[
\frac{d^3 u}{dx^3} = f(x), \quad u(0) = 0, \quad u(1) = 0, \quad u''(1) - u'(1) = 0
\]

and evaluate the integral \(\int g(x, s)f(s) \, ds\) for the solution of

\[
\frac{d^3 u}{dx^3} = 24x, \quad u(0) = 0, \quad u(1) = 0, \quad u''(1) - u'(1) = 0
\]

to confirm that you can obtain the exact solution, \(u(x) = x^4 - 9x^2 + 8x\).

Hint: if you are not getting this solution, you might check if you’ve accidentally swapped \(g(x, s)\) with \(g(s, x)\).
5. In lecture, we determined the Green’s function satisfying

\[
\frac{d^2 G}{dx^2} + G = \delta(x - s), \quad G(x = 0) = 0, \quad G(x = 1) = 0
\]

is

\[
G(x, s) = \begin{cases} 
\sin(s - 1) \sin(x) / \sin(1) & 0 \leq x < s \\
\sin(s) \sin(x - 1) / \sin(1) & s < x \leq 1 
\end{cases}
\]

Use this Green’s function to evaluate the integral representation for the solution \(u(x)\) (Haberman 9.3.51, p. 398) of the inhomogeneous problem

\[
\frac{d^2 u}{dx^2} + u = 2 \cos x \quad u(0) = 5, \quad u(1) = -2
\]

See Haberman, section 9.3.5, pages 397–399.