Before you start, please read the syllabus carefully.

1. For any ring $R$, prove that there exists a unique ring homomorphism from $\mathbb{Z}$ to $R$.

2. Find the number of group homomorphisms from $A$ to $B$, and the number of ring homomorphisms from $A$ to $B$ (Just numbers.)
   
   (a) $A = \mathbb{Z}$, $B = C_6$
   (b) $A = C_6$, $B = \mathbb{Z}$
   (c) $A = C_6$, $B = C_3$
   (d) $A = C_3$, $B = C_6$

3. Let $a$ be a rational number and define $f : \mathbb{Q}[x] \to \mathbb{Q}$ to be $f(g(x)) = g(a)$. Prove that $f$ is a ring homomorphism.

4. Let $R$ be a ring and $S$ be a subset of $R$. We define the ideal generated by $S$ to be the smallest ideal containing $S$.

   (a) Prove that $\langle 1 \rangle = R$.
   (b) Prove that $\langle S \rangle = \{ \sum_{i \leq N} r_i s_i t_i \mid r_i, t_i \in R, s_i \in S \}$.

5. For the ring of integers, denote $I = \langle 5 \rangle$, and $J = \langle 3 \rangle$

   (a) Prove that the ideal generated by $n$ is $\langle n \rangle = \{ kn \mid k \in \mathbb{Z} \}$.
   (b) Compute $I + J$, $I \cdot J$ and $I \cap J$.

6. For the ring of integers:

   (a) Prove that any subgroup of $(\mathbb{Z}, +)$ is cyclic.
       (Hint: Use the fact that if $\gcd(a, b) = d$, then $\exists s, t \in \mathbb{Z}$ such that $as + bt = d$).
   (b) Prove that any subgroup of $(\mathbb{Z}, +)$ is also an ideal of $(\mathbb{Z}, +, \times)$.