Permutation Group / Alternating Group

denoted $S_n$ denoted $A_n$

Recall,

$$S_n := \{ \sigma \mid \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \text{ that is bijective} \}$$

is composition.

Goal: Define $A_n$, describable $A_n$.

Normal Subgrp.

Recall we defined quotient ring for a ring $R$ by an ideal $I \subseteq R$, $R/I$.

to be $(\{ \text{equivalence classes of } \sim \} , + , \times)$.

Here $a \sim b$ in $R$ if

$$a - b \in I.$$  

Q: Can we do similar things for grp?

Last time, given $H \subseteq G$, we define $\sim_H$ to be

$$g_1 \sim g_2 \iff g_1^{-1}g_2 \in H$$

Can we define $\sim_{\text{grp operation}}$ on $\{ \text{equivalence classes} \}$

by picking representatives?

$$g \cdot H := \{ g \cdot h \mid h \in H \}$$
\( g_1 \cdot H \) and \( g_2 \cdot H \) are the representatives for the two cosets.

\[
\tilde{g}_1 \in G \quad \tilde{g}_2 \in G
\]

\[
(g_1 \cdot H) \cdot (g_2 \cdot H) = g_1 g_2 H
\]

We need to check whether this multiplication is well-defined?

\[
\tilde{g}_1 \cdot \tilde{g}_2 \in H \iff \tilde{g}_1 \sim g_1 \quad \tilde{g}_2 \sim g_2
\]

\[
\iff (\tilde{g}_1 \cdot \tilde{g}_2)^{-1} \cdot (g_1 \cdot g_2) \in H
\]

Now since \( \tilde{g}_1 \in g_1 \cdot H \) we have \( \tilde{g}_1 = g_1 \cdot h_1 \) for some \( h_1 \in H \)

Similarly, we have \( \tilde{g}_2 = g_2 \cdot h_2 \) for some \( h_2 \in H \)

\[
\iff (g_2 \cdot h_2)^{-1} \cdot (g_1 \cdot h_1)^{-1} \cdot g_1 \cdot g_2 \in H
\]

\[
\iff h_2^{-1} \cdot g_2^{-1} \cdot h_1^{-1} \cdot g_1 \cdot g_2 = h_2^{-1} \cdot g_2^{-1} \cdot h_1^{-1} \cdot g_1 \cdot g_2 \in H
\]

\[
\iff (g_1 \cdot h_1)^{-1} \cdot g_2 \cdot h_2 \cdot h_1^{-1} \cdot g_1 \cdot g_2 \in H
\]

Notice that we can choose any \( h_1 \in H \) and any \( g_2 \in G \) since we can choose any representative for \( g_1 \cdot H \) and \( g_2 \cdot H \) and we can choose any two cosets to do group operation.
In order to make the operation on cosets well-defined, we need \( H \leq g_2^{-1} H g_2 \) for any \( g_2 \in G \).

**Def (normal subgrp).** A subgroup \( H \leq G \) is normal if \( H \leq g^{-1} H g \) for any \( g \in G \).

**Rmk:** For finite groups, \( H \leq g^{-1} H g \iff H = g^{-1} H g \) since \( 1^{-1} H 1 = 1g^{-1} H g1 \).

We will write \( H \triangleleft G \) to imply \( H \) is normal in \( G \).

By the previous deductions, we show that:

**Lemma:** Given \( N \triangleleft G \), we have

\( \{ \text{cosets of } N \} \), \( \cdot \) forms a grp.

pf: Since \( N \triangleleft G \), the operation \( \cdot \) is well-defined.

The identity law / associative law / inverse law:

\[
(e \cdot H) \cdot (g \cdot H) = (g \cdot e) \cdot H = gH, \quad (g_1H)(g_2H)(g_3H) = (g_1H)(g_2Hg_3H)
\]

all follows from that of the original grp \( G \).

**Def (quotient grp) Given** \( N \triangleleft G \), then the grp

\( \{ \text{cosets of } N \}, \cdot \)

is called the quotient grp of \( G \) by \( N \). Denote it by \( G/N \).
Lagrange

Fact: \( |G/N| = \frac{|G|}{|N|} \neq [G : N] \)

Example: \( S_3 \) contains 6 elements.

Flip
\[ G = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \]

Rotation
\[ T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \]

Identity
\[ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \]

Q: 1) \( H_1 = <G> \)  How large is \( H_1 \)?
Is \( H_1 \) normal in \( G = S_3 \)?

2) \( H_2 = <T> \)  How large?
Normal?

Ans. 1) \( T^{-1} G T = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \neq H_1 \)

\( H_1 \) not normal

2) \([S_3 : H_2] = 2\).  So \( H_2 \) must be normal in \( S_3 \).

\( H_2 \ 6\cdot H_2 \quad 6H_2 = H_2 \ 6 \)
Lemma. Given \( H \leq G \), if \([G:H]=2\), then \( H \triangleleft G \).

pf: \( H \triangleleft G \iff gHg^{-1} = H \ \forall \ g \in G \) 
\[ \iff Hg = gH \ \forall \ g \in G \]

right coset left coset.
\[ \{ hg \mid h \in H \} \quad \{ gh \mid h \in H \} \]

But if \([G:H]=2\), the \( G = H \cup gH \) 
\[ g \in G, \quad G = H \cup H.g_g \]

so \( g,H = H.g_g \)
Actually \( \forall \ g \notin H \) we have \( gH = g.H \)

\[ H.g_g = H.g \]

so \( \forall g \in G, \quad gH = Hg \).

Another method for \( 2) \) is to check

\[ 6 \cdot 2 : 6 \in <2> \]

Lemma: \( \forall g, \quad gHg^{-1} = H \iff \)

\( \forall g_i, \quad g_iHg_i^{-1} = H \) where \( g_i \) are within a set of representatives for left cosets of \( H \).

(\( i.e. \ G = \bigcup_{i=1}^{k} g_iH, \ k = [G:H] \)).

Pf. Exercise.
This implies it is enough to check
\[ g^{-1} H g \subseteq \mathcal{C} \, . \]

After Class Remark:

The definition on normal subgroups require
\[ H \leq g^{-1} H g \quad \forall g \in G \]
notice that by multiplying on both sides by \( g \) and \( g^{-1} \)
\[ g H g^{-1} \leq H \, . \]
so \[ H \leq g^{-1} H g \quad \forall g \in G \quad \Rightarrow \quad H = g^{-1} H g \quad \forall g \in G \]
This does not require a "size" argument in the lecture.