Homework 9, Math 401

due on April 13, 2020

Before you start, please read the syllabus carefully.

1. Given \( n \geq 4 \).

   (a) Show that \( A_n \) can be generated by all 3-cycles (meaning that the smallest subgroup containing all 3-cycles is \( A_n \) itself).

   (b) Show that if \( H \triangleleft A_n \) is a normal subgroup, and \( H \) contains one 3-cycle, then \( H \) contains all 3-cycles.

2. We study \( G = A_4 \).

   (a) Determine all subgroups of \( A_4 \), and determine whether they are normal or not.

   (b) Write down all increasing sequences of subgroups \( G_0 = e \subset G_1 \subset G_2 \subset \cdots \subset G_n = G \) where \( G_i \triangleleft G_{i+1} \) and \( G_{i+1}/G_i \) is abelian.

**Answer:**

1) Aside from \( A_4 \) and \( \{e\} \), there are 4 subgroups isomorphic to \( C_3 \), \( \langle (123) \rangle \), \( \langle (124) \rangle \), \( \langle (134) \rangle \), \( \langle (234) \rangle \), they are not normal, e.g. \( (421)(123)(124) = (134) \). There are 1 subgroup isomorphic to \( C_2 \times C_2 \), it is \( \{(12)(34), (13)(24), (14)(23)\} \). It is normal since conjugation does not change cycle type, and all elements with such cycle types are in this subgroup. There are 3 subgroups of this order 4 normal subgroup, \( \langle (12)(34) \rangle \), \( \langle (13)(24) \rangle \), \( \langle (14)(23) \rangle \), they are not normal, e.g. \( (123)(12)(34)(321) = (23)(14) \). 2) \( \{e\} \subset \langle (12)(34) \rangle \subset \{ (12)(34), (13)(24), (14)(23) \} \subset G \).

Other options include replacing \( G_1 \) by \( \langle (13)(24) \rangle \) or \( \langle (14)(23) \rangle \), or simply drop \( G_1 \) in the above sequence. These are all the options since \( \{ (12)(34), (13)(24), (14)(23) \} \) is the only non-trivial normal subgroup of \( G \) and with quotient group \( C_3 \).

3. Let \( G \) be a finite group. Define a relation on \( G \): \( g_1 \sim g_2 \) iff there exists \( \sigma \in G \) such that \( \sigma g_1 \sigma^{-1} = g_2 \). Show that \( \sim \) is an equivalence relation. (The equivalence class is called conjugacy classes of \( G \).)

4. Let \( G \) be a finite group. Define a relation on subgroups of \( G \): \( H_1 \sim H_2 \) iff there exists \( \sigma \in G \) such that \( \sigma H_1 \sigma^{-1} = H_2 \). Show that \( \sim \) is an equivalence relation. (The equivalence class is called conjugacy classes of subgroups of \( G \).)

5. Let \( G \) be a finite group and \( N \triangleleft G \) be a normal subgroup and \( H \subset G \) be a subgroup.

   (a) We denote \( N \cdot H \) to be the subset \( \{ n \cdot h \mid n \in N, h \in H \} \) of \( G \). Show that \( N \cdot H \) is a subgroup.
(b) Show that \( N \cdot H = H \cdot N \).
(c) Show that \( N \cap H \) is a normal subgroup of \( H \).
(d) Show that \( N \cap H \) is normal, we have \( hN^{-1}N = N \). Therefore \( n_1h_1n_2h_2 = (n_1h_1n_2h_2)N \). Similarly, any element \( hN = hN^{-1} \cdot h \in N \cdot H \). Therefore the two sets are equal.

c) For any element \( h \in H \), we have \( hN \cap Hh^{-1} \cap hNh^{-1} = N \cap H \). Therefore \( N \cap H \) is normal in \( H \). d) Define the map \( f : H \to N \cdot H/N \) with \( f(h) = hN \). It is easy to check \( f \) is a group homomorphism (check here by yourself). The map \( f \) is surjective since by (b) any coset of \( N \) can be written as \( hN = hN \) for certain element \( h \in H \). The kernel of \( f \) is \( N \cap H \). By fundamental homomorphism for group, we have \( N \cdot H/N \simeq H/(N \cap H) \).

6. Let \( N \trianglelefteq G \) be a normal subgroup of \( G \). Show that if \( G \) is solvable, then \( G/N \) is solvable.

7. Given a finite group \( G \). Prove that if \( |G| = p \) and \( p \) is a prime number, then \( G \simeq C_p \), where \( C_p \) is the cyclic group with order \( p \).

8. Given a finite group \( G \). Prove that if \( |G| = p^n \) some prime power, then there exists \( g \in G \) with \( \text{ord}(g) = p \).

9. Let \( G \) be a finite group. Prove that if \( G/N_1 \) and \( G/N_2 \) are abelian, then \( G/(N_1 \cap N_2) \) is also abelian. (Therefore we can define \( G^{ab} \) to be the maximal quotient group \( G/N \) that is abelian, we call it the abelianization of \( G \).)

10. Denote \( [G, G] \) to be the smallest subgroup of \( G \) containing \( g_1g_2g_1^{-1}g_2^{-1} \) for all \( g_1, g_2 \in G \). Prove that \( G/[G, G] = G^{ab} \).