Homework 7, Math 401

due on March 16, 2020

Before you start, please read the syllabus carefully.

1. Determine the splitting field of \( f(x) = x^5 - 2 \in \mathbb{Q}[x] \). **Answer:** \( \mathbb{Q}^{\sqrt[5]{2}, 2^{1/5}} \)

2. Consider \( f(x) = x^8 - 1 \in \mathbb{Q}[x] \).
   
   (a) Determine the factorization of \( f(x) \) into product of irreducible polynomials over \( \mathbb{Q}[x] \). **Answer:** \( x^8 - 1 = (x^4 + 1)(x^2 + 1)(x + 1)(x - 1) \)

   (b) Determine the splitting field \( K \) of \( x^4 + 1 \in \mathbb{Q}[x] \). **Answer:** \( \mathbb{Q}[x]/(x^4 + 1) \simeq \mathbb{Q}[\zeta_4] \)

3. Denote \( K = \mathbb{Q}[\sqrt[5]{2} + \sqrt[5]{3}] \subset \mathbb{C} \).
   
   (a) Prove that \( K \subset F := \mathbb{Q}[\sqrt{2}, \sqrt{3}] \). **Answer:** It follows since \( \sqrt{2} + \sqrt{3} \in F \).

   (b) Determine \( [F : \mathbb{Q}] \). **Answer:** 4

   (c) Denote \( \alpha = \sqrt[5]{2} + \sqrt[5]{3} \). Prove that \( 1, \alpha, \alpha^2, \alpha^3 \) are linearly independent over \( \mathbb{Q} \). **Answer:** By computation \( \alpha^2 = 7 + 2\sqrt[5]{10} \) and \( \alpha^3 = 17\sqrt[5]{2} + 11\sqrt[5]{5} \). Suppose \( a + b\alpha + c\alpha^2 + d\alpha^3 = 0 \) with \( a, b, c, d \in \mathbb{Q} \), comparing the coefficients of \( 1, \sqrt[5]{2}, \sqrt[5]{3}, \sqrt[5]{10} \), we get \( a + 7c = 0 \), \( c = 0 \), \( 17d + b = 0 \), \( 11d + b = 0 \). Solving this set of linear equations, we get \( a = b = c = d = 0 \).

   (d) Prove that \( K = F \). **Answer:** They have the same degree.

   (e) Prove that \( \{1, \alpha, \alpha^2, \alpha^3\} \) is a basis for \( K \) as a vector space over \( \mathbb{Q} \). **Answer:** We have shown before that they are linearly independent. On the other hand, \( K = \mathbb{Q}[\alpha] \) therefore every elements in \( K \) are linear combinations of \( \{1, \alpha, \alpha^2, \alpha^3\} \).

   (f) Write \( \alpha^4 \) as a linear combination of \( \{1, \alpha, \alpha^2, \alpha^3\} \), i.e., find \( a, b, c, d \in \mathbb{Q} \) such that
      
      \[
      \alpha^4 = a + b\alpha + c\alpha^2 + d\alpha^3.
      \]
      
      **Answer:** By computation, \( \alpha^4 = 89 + 28\sqrt[5]{10} \). Again solving for \( a, b, c, d \) by comparing the coefficients of \( 1, \sqrt[5]{2}, \sqrt[5]{3}, \sqrt[5]{10} \), we get \( a + 7c = 89 \), \( 2c = 28 \), \( b = d = 0 \), which is \( a = -9 \), \( b = 0 \), \( c = 14 \), \( d = 0 \).

4. (a) Prove that \( g(x) = x^4 + 1 \) is reducible over \( \mathbb{F}_3 \). (Hint: find \( h(x) | g(x) \))

   (b) Determine the factorization of \( f(x) = x^9 - x \in \mathbb{F}_3[x] \) into irreducible polynomials.

   (c) Determine the splitting field of \( f(x) \) and its degree over \( \mathbb{F}_3 \).