Before you start, please read the syllabus carefully.

1. Consider $\mathbb{Z}[^{\sqrt{-1}}]$. As a set, it contains all elements in the form of $a + b\sqrt{-1}$ where $a$ and $b$ are in $\mathbb{Z}$. The addition and multiplication is defined as the same addition and multiplication in complex numbers. Prove that $\mathbb{Z}[^{\sqrt{-1}}]$ is a commutative ring.

2. Prove that $\mathbb{Z}[^{\sqrt{-1}}]$ is an integral domain.

3. Given a surjective ring homomorphism $\phi : A \to B$ between two commutative rings.
   (a) Denote $J$ to be an ideal of $B$. Prove that $\phi^{-1}(J) := \{x \in A \mid \phi(x) \in J\}$ is an ideal of $A$.
   (b) Prove that if every ideal of $A$ is principal, then every ideal of $B$ is principle.

4. Find all the ideals in the ring of
   (a) $\mathbb{Z}$
   (b) $F[x]$ where $F$ is a field
   (c) $\mathbb{Z}_p$ where $p$ is a prime
   (d) $\mathbb{Z}_{pq}$ where $p$ and $q$ are two different primes.
   (e) $\mathbb{Z}_{p^2}$ where $p$ is a prime.

5. Show that $\mathbb{Z}_5$ is a quotient ring of $\mathbb{Z}_{10}$ (equivalently, this means that $\mathbb{Z}_5$ is isomorphic to a quotient ring of $\mathbb{Z}_{10}$).

6. Given $R$ a commutative ring. Prove that $I \cdot J := \{ \sum_{1 \leq k \leq K} i_k \cdot j_k \mid i \in I, j \in J\}$ are still ideals of $R$ where $I$ and $J$ are both ideals of $R$.

7. For the ring of integers $\mathbb{Z}$, denote $I = \langle m \rangle$ and $J = \langle n \rangle$. You have seen in previous exercises that $I + J$ and $I \cap J$ and $I \cdot J$ are all ideals for the same ring $R$. Also you have seen that all ideals of $\mathbb{Z}$ are principle. Find the generator for the following ideal:
   (a) $I + J$
   (b) $I \cap J$
   (c) $I \cdot J$
   **Bonus:** Which ideal is bigger between $I \cap J$ and $I \cdot J$? Can you guess when $I \cap J = I \cdot J$ for the ring $\mathbb{Z}$?

8. Find all ring homomorphisms $\phi : \mathbb{Q}[x] \to \mathbb{Q}$.
9. Prove that $\phi_a : F[x] \rightarrow F$ by mapping $\phi_a(f(x)) = f(a)$ is a surjective ring homomorphism. Determine $\text{Ker}(\phi_a)$. Show that $F$ is a quotient ring of $F[x]$.

10. Given $I = \langle x^2 + 5 \rangle$ an ideal of $R = F[x]$. Determine $R/I$ as a set, i.e., determine all the equivalence classes mod $I$. 