Problem 1 : Iterated Integral

Compute the following iterated integral:

1. $\int_0^2 \int_{-1}^{x^2} \int_1^y xyz\,dz\,dy\,dx$
2. $\int_0^{\pi/2} \int_0^{\sin\theta} \int_0^{r\cos\theta} r^2\,dz\,dr\,d\theta$;

1. $\int_0^2 \int_{-1}^{x^2} x \cdot y \cdot \frac{z^2}{2} \Big|_{z=1}^{z=y} dy\,dx$

$= \int_0^2 \int_{-1}^{x^2} xy\left(\frac{y^2}{2} - \frac{1}{2}\right) dy\,dx$

$= \int_0^2 \left(\frac{x}{2} \cdot \frac{y^4}{4} - \frac{x}{2} \cdot \frac{y^2}{2}\right) \Big|_{y=-1}^{y=x^2} dx$

$= \int_0^2 \left(\frac{x^9}{8} - \frac{x^5}{4} - \frac{x}{8} + \frac{x}{4}\right) dx$

$= \frac{2^{10}}{80} - \frac{2^6}{24} + \frac{2^2}{16}$

2. $\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r^2 \cdot z \Big|_{z=0}^{z=r\cos\theta} dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r^3 \cos\theta \cdot dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \cos\theta \cdot \frac{r^4}{4} \Big|_{r=0}^{r=\sin\theta} d\theta$

$= \int_0^{\frac{\pi}{2}} \cos\theta \cdot \frac{\sin^4\theta}{4} d\theta \underset{du=\cos\theta\,d\theta}{\overset{u=\sin\theta}{=\!=}} \int_0^1 \frac{u^4}{4} du$

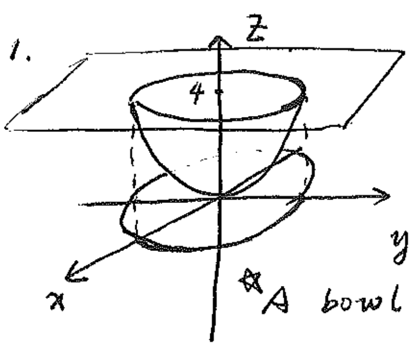
$= \frac{1}{20}$

Problem 2 : Describe Region

For all the region:
1. Sketch the region D;
2. Write the iterated integral on this region.

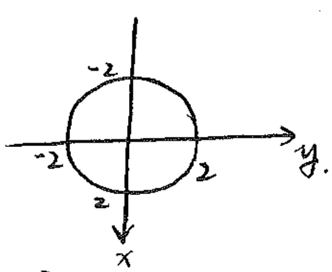
1. The region bounded by $z = x^2 + y^2$ and $z = 4$;
2. The region in the first octant bounded by $x + y + z = 9$, $2x + 3y = 18$ and $x + 3y = 9$.
3. The region bounded by $x^2 + y^2 = 1$ and $z = 0$, $z = 5$.
4. The region in the first octant bounded by $x^2 + y^2 = a^2$, and $z = x + y$.

1.



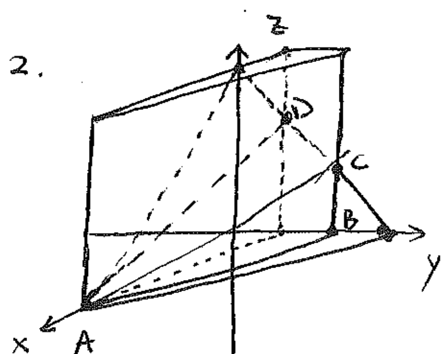
$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} dz\,dy\,dx$

Projection to xy-plane:



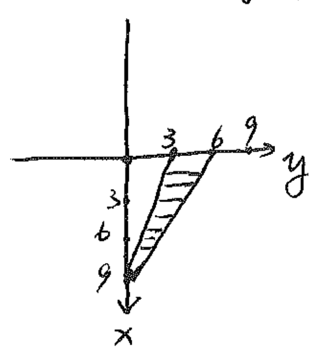
$\begin{cases} z = x^2 + y^2 \\ z = 4 \end{cases} \Rightarrow x^2 + y^2 = 4$

2.



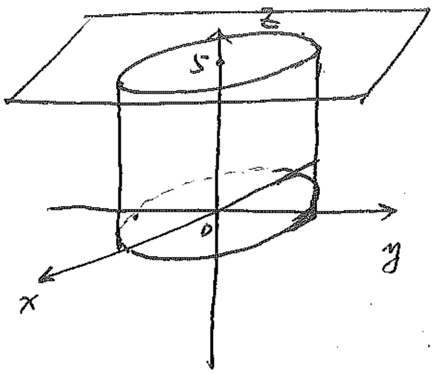
The shape is ABCD. Like a wedge.

Projection to xy-plane:



$\int_0^9 \int_{\frac{9-x}{3}}^{\frac{18-2x}{3}} \int_0^{9-x-y} dz\,dy\,dx$

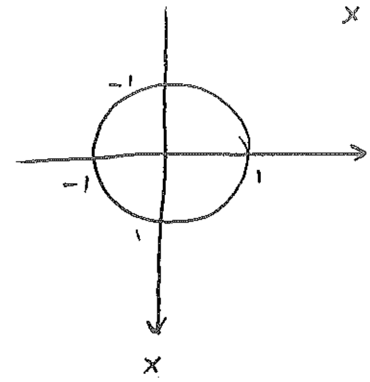
3.



A cylinder

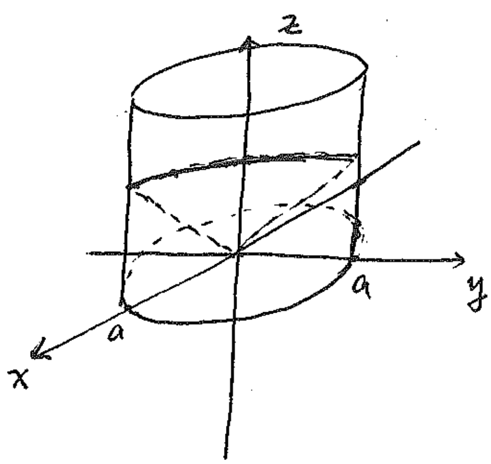
Projection:

$x^2+y^2=1$



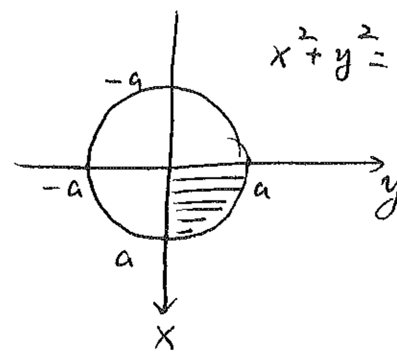
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{5} dz\,dy\,dx$$

4.



Projection:

$x^2+y^2=a^2$



$$\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{x+y} dz\,dy\,dx$$