Problem 1 : Tangent Vector and Tangent Line

Compute the tangent vector of following curves and write the parametrization of the tangent line at the given point:

1. $\begin{pmatrix} t^2 \\ t^3 \\ t^4 \end{pmatrix}$, $t = 1$

2. $\begin{pmatrix} 2\theta - 2\sin\theta \\ 2 - 2\cos\theta \end{pmatrix}$, $\theta = \pi$

1. $\vec{x}'(t) = \begin{pmatrix} 2t \\ 3t^2 \\ 4t^3 \end{pmatrix}$

$\vec{x}(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\vec{x}'(1) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

the line :

$\vec{y}(s) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

2. $\vec{x}'(\theta) = \begin{pmatrix} 2 - 2\cos\theta \\ 2\sin\theta \end{pmatrix}$

$\vec{x}(\pi) = \begin{pmatrix} 2\pi \\ 4 \end{pmatrix}$

$\vec{x}'(\pi) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

the line :

$\vec{y}(s) = \begin{pmatrix} 2\pi \\ 4 \end{pmatrix} + s \cdot \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

Problem 2: Computing Arc Length

Compute the arc length between given points and find the middle point on the arc.

1. $\begin{pmatrix} \cos \pi t \\ \sin \pi t \\ t \end{pmatrix}$ from $t = 0$ to $t = \pi$

2. $\begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix}$ from $t = 0$ to $t = \pi$

3. $\begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$ from $t = 0$ to $t = \pi$

1. $\vec{x}'(t) = \begin{pmatrix} -\pi \sin \pi t \\ \pi \cos \pi t \\ 1 \end{pmatrix}$

$\|\vec{x}'(t)\| = \sqrt{\pi^2 \sin^2 \pi t + \pi^2 \cos^2 \pi t + 1^2}$

$= \sqrt{\pi^2 + 1}$

$\int_0^{\pi} \sqrt{\pi^2+1}\, dt = \boxed{\sqrt{\pi^2+1} \cdot \pi}$

$\int_0^{t_0} \sqrt{\pi^2+1}\, dt = \sqrt{\pi^2+1} \cdot t_0 = \sqrt{\pi^2+1} \cdot \frac{\pi}{2}$

$\Rightarrow \boxed{t_0 = \frac{\pi}{2}}$

2. $\vec{x}'(t) = \begin{pmatrix} e^t \cdot (\cos t - \sin t) \\ e^t \cdot (\sin t + \cos t) \end{pmatrix}$

$\|\vec{x}'(t)\| = e^t \cdot \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}$

$= \sqrt{2} \cdot e^t$

$\int_0^{\pi} \sqrt{2} \cdot e^t\, dt = \sqrt{2} \cdot e^t \Big|_0^{\pi} = \boxed{\sqrt{2} \cdot (e^{\pi} - 1)}$

$\int_0^{t_0} \sqrt{2} \cdot e^t dt = \sqrt{2} \cdot (e^{t_0} - 1) = \frac{\sqrt{2} \cdot (e^{\pi} - 1)}{2}$

$\Rightarrow e^{t_0} = \frac{e^{\pi} + 1}{2} \Rightarrow \boxed{t_0 = \ln\left(\frac{e^{\pi}+1}{2}\right)}$

3. $\vec{x}'(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix}$ $\|\vec{x}'(t)\| = \sqrt{2 - 2\cos\theta}$

$= 2 \sin \frac{\theta}{2}$

$\int_0^{\pi} 2 \sin \frac{\theta}{2} d\theta = -4 \cos \frac{\theta}{2} \Big|_0^{\pi} = 4$

$\int_0^{t_0} 2 \sin \frac{\theta}{2} d\theta = \int_{t_0}^{\pi} 2 \sin \frac{\theta}{2} d\theta = \frac{4}{2}$

$\Rightarrow -4 (0 - \cos \frac{t_0}{2}) = 2 \Rightarrow \boxed{t_0 = \frac{2\pi}{3}}$

Problem 3: Curvature (Look at previous page for $\vec{x}'$ and $\|\vec{x}'\|$

Compute the curvature vector for the following curves:

1. $\begin{pmatrix} \cos \pi t \\ \sin \pi t \\ t \end{pmatrix}$

2. $\begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix}$

1. $\vec{T}(t) = \dfrac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = \dfrac{1}{\sqrt{\pi^2+1}} \cdot \begin{pmatrix} -\pi \sin \pi t \\ \pi \cos \pi t \\ 1 \end{pmatrix}$

$\vec{T}'(t) = \dfrac{1}{\sqrt{\pi^2+1}} \cdot \begin{pmatrix} -\pi^2 \cos \pi t \\ -\pi^2 \sin \pi t \\ 0 \end{pmatrix}$

$\vec{\kappa}(t) = \dfrac{\vec{T}'(t)}{\|\vec{x}'(t)\|} = \dfrac{1}{\pi^2+1} \cdot \begin{pmatrix} -\pi^2 \cos \pi t \\ -\pi^2 \sin \pi t \\ 0 \end{pmatrix}$

2. $\vec{T}(t) = \dfrac{1}{\sqrt{2}} \begin{pmatrix} \cos t - \sin t \\ \sin t + \cos t \end{pmatrix}$

$\vec{T}'(t) = \dfrac{1}{\sqrt{2}} \cdot \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \end{pmatrix}$

$\vec{\kappa}(t) = \dfrac{\vec{T}'(t)}{\|\vec{x}'(t)\|} = \dfrac{1}{\sqrt{2} \cdot e^t} \cdot \dfrac{1}{\sqrt{2}} \cdot \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \end{pmatrix}$

$= \dfrac{1}{2e^t} \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \end{pmatrix}$

Problem 4: Mixture

1. HW Problem 4 on Page 32
2. Consider a cable of radius r and length L which is to be wrapped along a spool of radius R. How long must the spool be so that the cable dose not overlap itself?

1. HW 4;

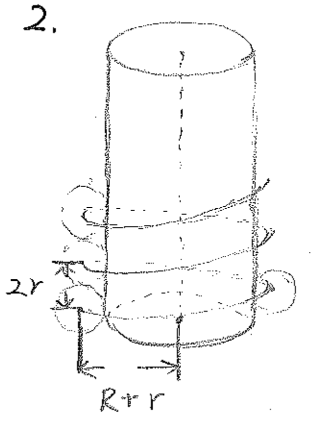
For any given $t = u$, we have tangent line :

$$\vec{y}_u(s) = \vec{x}(u) + s \cdot \vec{x}'(u) = \begin{pmatrix} u + s \\ u^2 + s \cdot 2u \\ u^3 + s \cdot 3u^2 \end{pmatrix}$$

let $u^3 + s \cdot 3u^2 = 0 \Rightarrow s = -\frac{u}{3} \Rightarrow$ intersection pt : $(\frac{2}{3}u, \frac{1}{3}u^2, 0)$

so $P(u) = (\frac{2}{3}u, \frac{1}{3}u^2, 0)$ it satisfy $y = \frac{3}{4}x^2$ so parabola.

2.



So the center of cable is a helix with parametrization

$$\vec{x}(\theta) = \begin{pmatrix} (R+r)\cos\theta \\ (R+r)\sin\theta \\ \frac{r}{\pi}\theta \end{pmatrix}$$

$$\int_0^{\theta_0} \|\vec{x}'(\theta)\| \, d\theta = \sqrt{(R+r)^2 + (\tfrac{r}{\pi})^2} \cdot \theta_0 = L$$

$$\Rightarrow \theta_0 = \frac{L}{\sqrt{(R+r)^2 + (\frac{r}{\pi})^2}}$$

So at least $\frac{\theta_0}{2\pi}$ round is needed,

spool should be $\frac{\theta_0}{2\pi} \cdot 2r$ at least.