Problem 1: Surface Integral Part 1: Area, Mass

Given the sphere \( \vec{x}(\theta, \phi) = \begin{pmatrix} R \sin \phi \cos \theta \\ R \sin \phi \sin \theta \\ R \cos \phi \end{pmatrix}, \ 0 \leq \phi \leq \pi, \ 0 \leq \theta \leq 2\pi, \)

1. What is the normal vector \( \vec{x}_\theta \times \vec{x}_\phi \)?

2. What is the area by surface integral?

3. What is the unit normal?

4. Suppose the density function \( \mu(\theta, \phi) = \cos^2 \theta \), compute the total mass of the this unit sphere?

5. What is the average \( z \)-coordinate for the upper sphere, i.e., \( z \geq 0 \)?

Problem 2: Surface Integral Part 2: Flux

Consider the same sphere as in part 1 with outward normal,

1. Given the vector field \( \vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \), compute the flux?

2. Given the vector field \( \vec{v} = \begin{pmatrix} y \\ -x \\ z \end{pmatrix} \), compute the flux?
Problem 3: Divergence Theorem
Consider the same question in part 2 using divergence theorem

1. Problem 2.1
2. Problem 2.2
3. Given the vector field \( \mathbf{v} = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} \), what is the flux?

Problem 4: Stokes Theorem
Consider the same sphere, and take \( \gamma \) to be the boundary of the upper sphere with counterclockwise condition,

1. Given the vector field \( \mathbf{F} = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix} \), use Stoke’s theorem to compute \( \oint_\gamma \mathbf{F} \cdot d\mathbf{s} \).

2. Consider \( \gamma \) is the triangle with vertex to be \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\), and \( \mathbf{F} = \begin{pmatrix} x + y^2 \\ y + z^2 \\ z + x^2 \end{pmatrix} \), what is \( \oint_\gamma \mathbf{F} \cdot d\mathbf{s} \)?