

THE 2-CLASS TOWER OF $\mathbb{Q}(\sqrt{-5460})$

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ABSTRACT. The seminal papers in the field of root-discriminant bounds are those of Odlyzko and Martinet. Both papers include the question of whether the field $\mathbb{Q}(\sqrt{-5460})$ has finite or infinite 2-class tower. This is a critical case that will either substantially lower the best known upper bound for \liminf of root-discriminants (if infinite) or else give a counter-example to what is often termed Martinet's conjecture or question (if finite). Using extensive computation and introducing some new techniques, we give strong evidence that the tower is in fact finite, establishing other properties of its Galois group en route.

1. INTRODUCTION

If K is a number field, then its root discriminant $rd(K)$ is an important invariant defined to be the $[K : \mathbb{Q}]$ th root of the absolute value of its discriminant. In this paper we are concerned with the multi-set of real numbers $rd(K)$ as K varies. Its smallest values are easily computed to be $1, \sqrt{3}, 2, \sqrt{5}, \dots$ (with multiplicity one). For large enough C there are infinitely many number fields with $rd(K) \leq C$. The main question is to find the smallest such C , in other words $C_0 := \liminf rd(K)$. It has long been known [18] that $C_0 \geq 4\pi e^\gamma$ and that under GRH $C_0 \geq 8\pi e^\gamma \approx 44.76$. As for upper bounds, the best obtained so far [10] gives $C_0 < 82.2$.

Upper bounds are typically obtained as follows. If L/K is an unramified extension, then $rd(L) = rd(K)$. It follows that if K has infinitely many unramified extensions, or equivalently if the Galois group of its maximal unramified extension is infinite, then $C_0 \leq rd(K)$. In practice it is hard to compute this Galois group for a given K . We can usually say much more about the Galois group of the maximal unramified p -extension of K , meaning the compositum of all unramified Galois extensions of p -power degree, for some prime p . This extension is also called the *p -class tower* of K and its Galois group will be termed the *p -tower group*. A few recent papers [5], [6], [17], have studied the 2-tower groups of several quadratic fields with root discriminant less than 82.2 in attempts to improve upon the upper bound for C_0 , but in each case so far the group has turned out to be finite. One of the few unresolved instances remaining is that of $\mathbb{Q}(\sqrt{-5460})$, a case that was mentioned as important all the way back in Odlyzko's seminal paper [19]. It is the smallest imaginary quadratic field whose 2-class group has rank 4, which implies that the 2-tower group is a pro-2 group with 4 generators. This considerably increases the complexity of the problem of determining whether this group is finite or infinite, and the current paper is the first serious attempt to address these complications. If its 2-class tower turns out to be infinite, then the known upper bound for C_0 will be lowered to $\sqrt{5460} \approx 73.89$.

In [15], Martinet noted that imaginary quadratic fields with 2-class group of rank > 4 have infinite 2-class tower by Golod-Shafarevich [9]. He raised the question of whether those of rank equal to 4 always have infinite 2-class tower. This has inspired a lot of recent work. Benjamin [1],[2],[3], Sueyoshi [23],[24],[25], Mouhib [16], and Wang [26] have established that

the tower is infinite in many cases, most of which have nonzero 4-rank. The case of zero 4-rank, which includes that of $\mathbb{Q}(\sqrt{-5460})$, is the most challenging, and the case at hand is as yet still open.

In fact, below, we give evidence that the 2-class tower of $K = \mathbb{Q}(\sqrt{-5460})$ is finite (but very large). This runs contrary to what is usually conjectured [15],[26]. We show that its Galois group G_K has 4 generators and exactly 5 relators and compute information on its maximal quotient $Q_c(G_K)$ of nilpotency class $c = 2, 3$. In [15], Martinet noted that he knew of no finite 2-group with 4 generators and 5 relators. Our method yields millions of such groups. It is seen that almost all of our candidates for G_K are finite and that often, for a given c and candidate for $Q_c(G_K)$, all corresponding candidates for G_K are finite.

The key empirical observation is that for each candidate for $Q_3(G_K)$ all its descendants G have the property that G^8 is abelian and of finite, bounded index. Once checked in its entirety, this will imply that G_K is finite.

We make extensive use of the computer algebra systems Magma [4] and Pari Gp.

2. 2-TOWER GROUPS OF IMAGINARY QUADRATIC FIELDS

Let K be an imaginary quadratic field and let G be its 2-tower group. Then G is a (topologically) finitely presented pro-2 group. Let $d(G)$ and $r(G)$ denote its generator and relator rank respectively. By Burnside's basis theorem, $d(G) = d(G^{ab})$, where G^{ab} is the maximal abelian quotient of G , which here is isomorphic to the 2-class group of K . As for the relator rank, by [21], $r(G)$ is known to equal $d(G)$ or $d(G) + 1$.

Moreover, by Golod-Shafarevich [9], if G is finite, then $r(G) > d(G)^2/4$. Combining the last two statements gives that if $d(G) \geq 5$, then G is infinite.

Let K denote $\mathbb{Q}(\sqrt{-5460})$ and G_K its 2-tower group. This paper will provide lengthy information regarding the structure of G_K . The first such information is that $G_K^{ab} \cong (\mathbb{Z}/2)^4$, which we denote $[2, 2, 2, 2]$. (In general, the group $\mathbb{Z}/2^{k_1} \times \dots \times \mathbb{Z}/2^{k_a}$ will be denoted $[2^{k_1}, \dots, 2^{k_a}]$ for short). It follows that $d(G_K) = 4$ and so $r(G_K) = 4$ or 5 . There are known to exist finite 2-groups with $d(G) = 4$ and $r(G) = 5$ (the smallest [11] has order 2^{14} - our method below yields millions of 4-generator 5-relator 2-groups).

3. INGREDIENTS REGARDING p -GROUP THEORY

If G is a finite p -group, then its p -central series of subgroups is defined by $P_0(G) = G, P_{n+1}(G) = [G, P_n(G)]P_n(G)^p (n \geq 0)$. If G is a finitely generated pro- p group, then we take the closed subgroups generated by these expressions. In particular, $P_1(G)$ is the Frattini subgroup of G . If $P_c(G) = \{1\}$, but $P_{c-1}(G) \neq \{1\}$, then we say that G has p -class c . The unique (up to isomorphism) p -group with d generators and p -class 1 is the elementary abelian p -group $(\mathbb{Z}/p)^d$. The quotient $Q_c(G) := G/P_c(G)$ is the maximal quotient of G of p -class $\leq c$ and is finite. If G has abelianization of exponent p (as is the case for the Galois group of interest to us), then in fact all the factors in its lower central series have exponent p , and so its p -central series coincides with its lower central series and its p -class is the same as its nilpotency class [20, section 5.2.5].

Given p and d , the set of (isomorphism classes of) finite d -generator p -groups forms a rooted tree with root $(\mathbb{Z}/p)^d$. An edge joins G to H if G has p -class c and $H \cong Q_{c-1}(G)$. Thus, the vertices at length k from the root are precisely the groups of p -class $k+1$. O'Brien has provided an algorithm [18] that, given a p -group G of p -class c , yields all its neighbors

of p -class $c + 1$. These are called the *immediate descendants* (or *children*) of G . Some groups (such as the quaternion group of order 8) have no descendants and are called *terminal*. An equivalent characterization is that their nuclear rank is 0. The nuclear rank (defined below) of a group is loosely correlated with how many children it has.

Let G be a finite d -generator p -group of p -class c . It can be presented as F/R , where F is a free group on d generators. Let $R^* = [R, F]R^p$, a characteristic subgroup of R . The group $G^* := F/R^*$ is a finite group called a p -covering group of G and the quotient R/R^* is the p -multiplier, a finite elementary abelian group. The p -multiplier rank of G is the rank of R/R^* . The nucleus of G is $P_c(G^*) = P_c(F^*)R^*/R^* \leq R/R^*$, and its rank defines the *nuclear rank* of G . These are all important players in O'Brien's algorithm. In particular, every child of G arises as a quotient of G^* . His algorithm sorts out isomorphism classes of such quotients F/M , where M/R^* is a proper subgroup of the p -multiplier that supplements the nucleus.

In [5], we note the following useful proposition:

Proposition 3.1. *If G is a finite p -group, then for any c*

$$p\text{-multiplier rank}(Q_c(G)) - \text{nuclear rank}(Q_c(G)) \leq r(G)$$

If G is an infinite d -generator pro- p group, then it is determined by its quotients $Q_c(G)$ ($c = 1, 2, \dots$), which define an infinite path through the above tree. We can therefore locate G as an end of this tree. The 2-tower group G_K of $K := \mathbb{Q}(\sqrt{-5460})$ is somewhere within the O'Brien tree (if finite) or at an end (if infinite). We will successively search for $Q_c(G_K)$ ($c = 1, 2, \dots$) and hence G_K within this tree. Our main tool is to prune the tree using information about finite unramified field extensions L of $\mathbb{Q}(\sqrt{-5460})$. Such a field L is the fixed field of an open subgroup H of G_K and the 2-class group of L tells us H^{ab} .

4. BEGINNING THE SEARCH

Let G_K be the 2-tower group of $\mathbb{Q}(\sqrt{-5460})$. As noted above, $Q_1(G_K) \cong (\mathbb{Z}/2)^4$. We next determine $Q_2(G_K)$ and simultaneously show that $r(G_K) = 5$. We know that $Q_2(G_K)$ is one of the children of $(\mathbb{Z}/2)^4$. Call it Q . There are 15170 such children. First, note that $Q^{ab} \cong [2, 2, 2, 2]$. Only 7851 of the children satisfy this. Applying proposition 3.1, using $r(G_K) \leq 5$ reduces us to 4722 possibilities for Q .

Next, we can use the abelianizations of low index subgroups obtained by computing 2-class groups of unramified extensions of small degree, as indicated at the end of the last section.

Proposition 4.1. *G_K has 15 subgroups of index 2. Their abelianizations are $[2, 4, 4]$ (8 times), $[2, 4, 8]$ (2 times), $[4, 4, 4]$ (1 time), $[2, 2, 2, 4]$ (2 times), and $[2, 2, 4, 4]$ (2 times).*

This still does not cut us down to one possibility for $Q_2(G_K)$, but work of Koch from 1964 [12], building on that of Fröhlich [8], does give an explicit presentation for $Q_2(G_K)$.

Theorem 4.2. *The Galois group of the maximal unramified 2-extension of nilpotency class 2 of $\mathbb{Q}(\sqrt{-5460})$, $Q_2(G_K)$, is the 2-class 2 quotient of the pro-2 group of order 2^9 and presentation*

$$\langle a, b, c, d \mid a^{-2}(d, c), b^{-2}(d, a)((d, b), b), c^{-2}(b, a)((d, b), b), d^{-2}(c, a)(d, a)(d, b), (b, c) \rangle$$

This group has p -multiplier rank 11 and nuclear rank 6. Proposition 3.1, together with [21], then implies the following.

Corollary 4.3. *The relator rank of the 2-tower group of $\mathbb{Q}(\sqrt{-5460})$ is 5.*

The large nuclear rank suggests that $Q_2(G_K)$ has many children, and indeed it has 151501 of them. Of these 32768 satisfy Propositions 3.1 and 4.1. This is still a huge number to handle and unfortunately there is no extension of Theorem 4.2 to higher nilpotency class and so we introduce some new techniques.

5. ABELIANIZATIONS OF LARGER INDEX SUBGROUPS

We can also compute the 2-class group for every unramified degree 4 extension of K , which then tells us the lattice of subgroups of G_K of index at most 4, together with their abelianizations.

Proposition 5.1. *Up to conjugacy, G_K has 51 subgroups of index 4. Their abelianizations are $[2, 4, 4]$ (2 times), $[2, 4, 8]$ (8 times), $[4, 4, 8]$ (3 times), $[2, 2, 2, 4]$ (2 times), $[2, 2, 4, 4]$ (14 times), $[2, 2, 4, 8]$ (9 times), $[2, 2, 4, 16]$ (5 times), $[2, 2, 8, 8]$ (1 time), $[2, 4, 4, 4]$ (3 times), $[2, 4, 4, 8]$ (1 time), $[2, 2, 2, 2, 4]$ (1 time), $[2, 2, 2, 4, 4]$ (1 time), $[2, 2, 2, 8, 8]$ (1 time).*

We also recorded lattice data, i.e. which index 2 subgroups contain which index 4 subgroups. Note that there are 5 conjugacy classes of subgroups that have a unique abelianization. We call these *critical* subgroups of index 4. We computed the abelianizations of their maximal subgroups (using pari gp to compute class groups of degree 16 extensions of \mathbb{Q}). This yielded the following.

Proposition 5.2. *As regards the 5 critical subgroups of index 4,*

- 1) *the unique one with abelianization $[2, 2, 8, 8]$ has index 2 subgroups with abelianizations $[2, 2, 8, 16]$ (10 times), $[4, 4, 8, 16]$ (2 times), $[2, 2, 4, 4, 16]$ (1 time), $[2, 2, 2, 8, 16]$ (1 time), $[2, 2, 2, 2, 8, 16]$ (1 time);*
- 2) *the unique one with abelianization $[2, 4, 4, 8]$ has index 2 subgroups with abelianizations $[4, 8, 8]$ (4 times), $[2, 2, 8, 8]$ (4 times), $[2, 4, 4, 8]$ (4 times), $[2, 2, 4, 4, 8]$ (1 time), $[2, 2, 4, 4, 16]$ (1 time), $[2, 2, 2, 4, 8, 8]$ (1 time);*
- 3) *the unique one with abelianization $[2, 2, 2, 2, 4]$ has index 2 subgroups with abelianizations $[2, 2, 2, 8]$ (4 times), $[2, 2, 4, 4]$ (6 times), $[2, 2, 4, 8]$ (16 times), $[2, 2, 2, 2, 4]$ (2 times), $[2, 2, 2, 4, 8]$ (1 time), $[2, 2, 2, 8, 8]$ (1 time), $[2, 2, 4, 4, 8]$ (1 time);*
- 4) *the unique one with abelianization $[2, 2, 2, 4, 4]$ has index 2 subgroups with abelianizations $[2, 2, 8, 8]$ (8 times), $[2, 4, 4, 8]$ (6 times), $[4, 4, 4, 4]$ (2 times), $[2, 2, 2, 4, 8]$ (8 times), $[2, 2, 4, 4, 4]$ (2 times), $[2, 2, 4, 4, 16]$ (1 time), $[2, 2, 2, 2, 4, 4]$ (1 time), $[2, 2, 2, 2, 8, 8]$ (2 times), $[2, 2, 2, 4, 8, 8]$ (1 time);*
- 5) *the unique one with abelianization $[2, 2, 2, 8, 8]$ has index 2 subgroups with abelianizations $[2, 2, 8, 8]$ (4 times), $[2, 2, 8, 16]$ (6 times), $[2, 4, 8, 8]$ (8 times), $[2, 4, 8, 16]$ (4 times), $[2, 2, 2, 8, 16]$ (4 times), $[2, 2, 4, 4, 8]$ (1 time), $[2, 2, 4, 8, 8]$ (1 time), $[2, 2, 4, 16, 16]$ (2 times), $[2, 2, 2, 4, 8, 8]$ (1 time);*

6. CAPITULATION

The index 4 abelianization information from Section 5 cuts the possibilities down from 32768 to 4096 candidates for $Q_3(G_K)$. Another technique we can usefully employ is capitulation. If H is a finite index subgroup of G_K , this concerns the transfer map $\varphi : G_K^{ab} \rightarrow H^{ab}$.

Let H be a finite index subgroup of G_K corresponding to extension L of K . Then H is the 2-tower group of L . As noted earlier, Class Field Theory gives an isomorphism between G_K^{ab} and the 2-class group of K and between H^{ab} and the 2-class group of L . Consider the following commutative square, where the upper horizontal map is the transfer map and the lower horizontal map is given by sending integral ideal $\alpha \subset O_K$ to the ideal class of αO_L :

$$\begin{array}{ccc} G_K^{ab} & \xrightarrow{\varphi} & H^{ab} \\ \downarrow iso & & \downarrow iso \\ Cl_2(K) & \xrightarrow{\alpha O_L} & Cl_2(L) \end{array}$$

The vertical maps are all isomorphisms, and so the kernel is the same for both the upper and lower horizontal maps. The strategy is to compute the kernel of the lower map using Class Field Theory and compare it with a computation using group theory of the kernel of the upper map.

On the group side, however, we only have some candidate quotient Q of G_K of 2-class 3 to work with. Let \bar{H} denote the image of H in Q . Consider the commutative square:

$$\begin{array}{ccc} Q^{ab} & \xrightarrow{\phi} & \bar{H}^{ab} \\ \uparrow & & \uparrow \\ G_K^{ab} & \xrightarrow{\varphi} & H^{ab} \end{array}$$

In general, it is difficult to determine the kernel of ϕ , but in our situation all 4096 children have the same abelianization as G_K^{ab} , and the abelianization of each index 2 subgroup matches the correct field data, and so, if H has index 2 in G_K , both the vertical maps are isomorphisms. We save those children where the group-theoretical kernel matches the true (number-theoretical) data. In fact, there could only be different kernels of ϕ for the 13th and 14th subgroups (as listed by Magma) of index 2. Saving the cases with a match reduces us to 512 possibilities for $Q_3(G_K)$. These all have order 2^{15} and nuclear rank 5 or 6 (and so have many children).

7. NILPOTENCY CLASS 3 QUOTIENT OF G_K

So far we have 512 candidates for $Q_3(G_K)$. We will now explain how extensive calculations reduce us to just 2 possibilities (the 14th and 52nd candidates in the list produced by Magma as described above). The strategy is to use the abelianizations of index 4 subgroups of G_K and of index 8 subgroups contained inside critical subgroups of index 4, i.e. Propositions 5.1 and 5.2. The corresponding abelianizations of the 512 candidates for $Q_3(G_K)$ are all consistent with this data, but for some of them all their children or grandchildren fail. For example, 384 of them (the last 384 in Magma's list) can be eliminated by using the following criterion of Nover [5].

Proposition 7.1. *Let G be a pro- p group, N a finite index normal subgroup, and V a word. Assume $P_c(G) \leq N$, if $[G/P_c(G) : V(N/P_c(G))] = [G/P_{c+1}(G) : V(N/P_{c+1}(G))]$, then $(N/P_c(N))/V(N/P_c(G)) \simeq N/V(N)$.*

We apply this with N the 5th critical subgroup. For those 384 groups, it has abelianization $[2, 2, 2, 4, 8]$. It turns out that for all their children too, this subgroup has abelianization $[2, 2, 2, 4, 8]$. Applying Nover's criterion, this subgroup will have abelianization $[2, 2, 2, 4, 8]$ for every descendant and so will never reach its true value of $[2, 2, 2, 8, 8]$. This eliminates those 384 groups.

By further use of low index abelianizations, we conjecturally get down to the following two possibilities for $Q_3(G_K)$. The point is that whereas each of our 128 groups typically has tens of thousands of children, there are much more limited possibilities for the abelianizations of the index ≤ 4 subgroups and of the index 8 subgroups contained inside the critical subgroups. In fact, each of the 128 candidate groups has either 2, 4, or 8 such possible lists of abelianizations. For a given group G , this set of lists can be found in practice by using a Magma program that randomly takes quotients of the p -covering group of G to produce children of G . Since each list arises equally often, we expect only to need at most about $8 \ln(8) \approx 17$ such quotients.

Likewise, we can produce such lists for further descendants of the 128 groups and compare them with the true list as given in Propositions 5.1 and 5.2. We find that matches apparently only arise for descendants (in fact grandchildren) of 2 of the 128 candidates. If Nover's lemma were true for non-normal subgroups N , then this would yield a proof - as things stand, we have plenty of evidence for the following conjecture.

Conjecture 7.2. *The Galois group of the maximal unramified 2-class 3 2-extension of $\mathbb{Q}(\sqrt{-5460})$, $Q_3(G_K)$, is the 3-class 2 quotient of the pro-2 group with presentation*

$$(1) \quad \langle a, b, c, d \mid a^{-2}(d, c), b^{-2}(d, a)((d, b), b), c^{-2}(b, a)((d, b), b), \\ d^{-2}(c, a)(d, a)(d, b)(b, a, a)(c, a, a)(d, a, a), (b, c) \rangle$$

or

$$(2) \quad \langle a, b, c, d \mid a^{-2}(d, c), b^{-2}(d, a)((d, b), b), c^{-2}(b, a)(b, a, d)(c, a, a)((d, b), b), \\ d^{-2}(c, a)(d, a)(d, b)(b, a, a)(b, a, d), (b, c) \rangle.$$

Each has order 2^{15} and nuclear rank 6.

As for understanding quotients of higher nilpotency class, we need to introduce more tools in the next two sections.

8. MORIBUND p -GROUPS

We call a p -group *moribund* if all its descendants are finite, in other words if there is no infinite path in the O'Brien tree starting at the given group. If our candidates for $Q_3(G_K)$ were moribund, then we would conclude that G_K must be finite. This is not true for them, but holds for many of their descendants. There is a simple test for a p -group to be moribund.

Theorem 8.1. *Suppose that G is a finite p -group. Let G_1 be its p -covering group, G_2 the p -covering group of G_1 , and so on. If there exists c such that the G_c has nuclear rank 0, then G is moribund.*

This test is used to avoid the need to compute huge numbers of descendants and to show that if G_K is infinite, then it must lie in certain areas of the O'Brien tree. For instance, there are 224 moribund groups among the original 512 candidates for $Q_3(G_K)$.

9. NILPOTENCY CLASS > 3 QUOTIENTS OF G_K

We have made extensive investigation of the descendants of the 128 candidates for $Q_3(G_K)$ and in particular of the two groups in Conjecture 7.2. For those two candidates we restrict to descendants satisfying Propositions 5.1 and 5.2, which is a moot restriction for large enough 2-class. We have found millions of such groups, with order as high as 2^{81} and nilpotency class as high as 14.

The most convincing evidence that G_K is finite comes from the following observation. If G is a 2-group, we define G^n to be the subgroup of G generated by all n th powers. If G is a pro-2 group, then G^n denotes the closed subgroup generated by all n th powers.

Conjecture 9.1. *Let G be any descendant of one of the 128 candidates for $Q_3(G_K)$. Then G^8 is of index at most 2^{40} and is abelian.*

For instance, the group G of order 2^{81} , mentioned above, has G^8 of index 2^{35} and abelian of rank 33. Nover's lemma can be used to show that if the index of the subgroup generated by 8th powers stabilizes when moving from a group to a child of that group, then the index is the same for all further descendants. The main consequence of the above conjecture is the following.

Corollary 9.2. *Conjecture 9.1 implies that G_K is finite.*

Proof. Suppose G_K is infinite. Then by Conjecture 9.1, for the sequence of its 2-class quotients, the index of the subgroup generated by 8th powers has to stabilize and the subgroup is abelian. In that case G_K has an infinite abelian subgroup of finite index. The corresponding number field has infinite 2-class group, which is impossible. \square

In fact the 128 candidates for $Q_3(G_K)$ fall into two classes, the first of which has considerably simpler descendants (for instance, this includes all moribund cases) and for any of these it appears that G^4 is abelian of index at most 2^{11} . This is not true for all descendants of the 2 candidates in Conjecture 7.2, both of which fall into the second class. It is, however, enlightening to investigate descendants of groups in the first class. For instance, we found the following infinite descendant of the 1st of the 128 candidates.

Example 9.3. *The 1st of the 128 groups (as listed by Magma) has an infinite descendant. This pro-2 group has presentation*

$$(3) \quad \langle a, b, c, d \mid a^{-2}(d, c), b^{-2}(d, a)(d, b, b)(b, a, a, c), \\ c^{-2}(b, a)(d, b, b)(b, a, a, c), d^{-2}(c, a)(d, a)(d, b), (b, c) \rangle$$

For this group G , the subgroup G^4 is abelian of index 2^{11} . Taking the center of the centralizer of G^4 yields an abelian normal subgroup A of index 2^8 with $A \cong [2, 4, 4, 8] \times (\mathbb{Z}_2)^4$. It follows that G has a free abelian normal subgroup of finite index and rank 4. In fact G is a 2-adic pre-space group, as studied in the solution of the coclass conjecture [13],[22].

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