Problem 1: Tangent Plane/Linear Approximation

For the following surfaces:
1) compute the partial derivatives;
2) write up the differential $df$ (i.e. linear approximations);
3) find out the tangent plane at the given point;
4) what is the normal vector of the plane you find out.

1. $z = xy^2$; $x = 2, y = 1$

2. $z = \frac{xy}{x+y}$; $x = 3, y = 1$

3. $x^2 + y^2 + z^2 = 3$; $x = 1, y = 1, z = 1$; $x = 1, y = 1, z = -1$

4. $\vec{n} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

Problem 2: Chain Rule

Firstly use chain rule to compute $\frac{df}{dt}$, and then evaluate $\frac{df}{dt}$ at given point: For points $(1, 1, -1)$

1. $f(x, y) = x^2y^3 + x^3y^2$; $x(t) = t^2 + t, y(t) = e^t; t = 0$

2. $f(x, y) = x^2 + y^2$; $x(t) = \cos t, y(t) = 2\sin t; t = \frac{\pi}{2}$

3. $f(x, y, z) = xyz$; $x(t) = \ln t, y(t) = e^t, z(t) = \frac{1}{t}; t = 2$

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}
\]

\[
= (2xy^3 + 3x^2y^2)(2t+1) + (3y^2x + 2yx^3). e^t
\]

\[
= \left. \frac{df}{dt} \right|_{t=0} = 0
\]

2. $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} + \frac{dy}{dy} \frac{dt}{dt} = 2y(1 - \sin t) + 2y(2\cos t)$

\[
\frac{dy}{dt} \bigg|_{t=\frac{\pi}{2}} = 2 \cdot 0 \cdot 1 + 2 \cdot 2 \cdot 0 = 0
\]

(because $x(\frac{\pi}{2}) = 0, y(\frac{\pi}{2}) = 2$)

3. $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} + \frac{dy}{dy} \frac{dy}{dt} + \frac{dy}{dz} \frac{dz}{dt} = yx \cdot \frac{1}{t} + xz \cdot e^t + xy \cdot \frac{1}{t^2}$

\[
\left. \frac{dy}{dt} \right|_{t=2} = e^2 \cdot \frac{1}{2} \cdot \frac{1}{2} + \ln 2 \cdot \frac{1}{2} \cdot e^2 + \ln 2 \cdot e^2 \cdot \frac{1}{4} = e^2 \cdot \frac{1}{2} \cdot \ln 2
\]
Problem 3: Gradient

Compute the gradient of the following functions
then determine that at the given point, in which direction the function increases the fastest,
and in which direction the function decreases the fastest, and in which direction the function remains the same?

1. \( f(x, y) = x^2 + 3xy^2; \ x = 1, y = 1 \)
2. \( f(x, y) = 100 - x^2 - 3y^3; \ x = 2, y = 1 \)

1. \( \nabla f = \begin{pmatrix} 2x + 3y^2 \\ 6xy \end{pmatrix} \quad \nabla f \bigg|_{(1,1)} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \)

so in direction \( \begin{pmatrix} 5 \\ 6 \end{pmatrix} \) \( f \) increases the fastest,
in direction \( \begin{pmatrix} -5 \\ -6 \end{pmatrix} \) \( f \) decreases the fastest,
in direction \( \begin{pmatrix} 4 \\ 9 \end{pmatrix} \), \( \begin{pmatrix} -4 \\ -9 \end{pmatrix} \) increases
in direction \( \begin{pmatrix} 9 \\ 4 \end{pmatrix} \), \( \begin{pmatrix} -9 \\ -4 \end{pmatrix} \) remains the same.

Problem 4: Tangent Plane Revisited

We learned that: given \( f(x, y, z) \), the level set is a surface, and the gradient is perpendicular to the tangent plane of the level set. Please use this idea to compute the tangent plane of the following surface:

1. \( x^2 + 3y^2 + z^3 = 5; \ x = 1, y = 1, z = 1 \)
2. \( x^2 + y^2 + z^2 = 1; \ x = 1, y = 0, z = 0 \)

1. Consider the surface as level set at \( 0 \) of
\( f(x, y, z) = x^2 + 3y^2 + z^3 - 5 \)
then. \( \nabla f = \begin{pmatrix} 2x \\ 6y \\ 3z^2 \end{pmatrix} \quad \nabla f \bigg|_{(1,1,1)} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \)
is the normal vector of tangent plane.

so that the plane is \( \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \left( \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} \right) = 0 \quad \iff \quad 2x + 6y + 3z = 11 \)

2. Consider the surface as level set at \( 0 \) of
\( f(x, y, z) = x^2 + y^2 + z^2 - 1 \)
\( \nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \quad \nabla f \bigg|_{(1,0,0)} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \)
is the normal vector of tangent plane.
so the plane is \( \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \left( \begin{pmatrix} x-1 \\ y-0 \\ z-0 \end{pmatrix} \right) = 0 \quad \iff \quad x = 1 \)