This coursepack serves as our text for the course, listing all learning objectives, providing required practice, and linking to free resources for reference and additional problems. **You are responsible for all coursepack problems: complete daily after lecture** to help solidify your understanding before the more challenging lab applications. Check your work using the solutions. For the best learning experience, try answering the questions without referring to your class notes first, then refer to your notes and the linked references, then chat with a classmate, and if you are still stuck, then seek assistance in Office Hours, the Help Room, and Ed Discussion. The coursepack may be updated during the semester. Please let me know if you spot any typos.

Reference materials

We do not have an official textbook: all content will be provided through lecture notes, lab worksheets, and homework (both the daily practice that appears here and weekly problem sets). However, it can often be helpful to see alternate presentations of the material, or to have a searchable source for reference, including for examples and practice problems.

Traditional textbooks

The following two textbooks have been used often in the single-variable Calculus sequence at Duke. Both are excellent and freely available for loan at both Perkins and Lily libraries.

- *Calculus* by Deborah Hughes-Hallett, et.al. (Referred to below as 'Hughes-Hallett Calculus'.) See the library listings for the 6th ed. (ISBN 9780470888643) and 7th ed. (ISBN 9781119444190)
- *Calculus: concepts and contexts* by James Stewart. See the library listing for the 4th ed. (ISBN 9780495557425). I don't reference this textbook specifically below, but it should be straightforward to find the relevant sections on each topic.

Open access electronic resources

The following resources are all open access and/or open source, available in html and/or PDF formats. Some of the practice problems below may be drawn from these sources as credited. If you find other open resources particularly helpful, please share!

- Active Calculus by Matt Boelkins, et al.
- OpenStax Calculus, Vol 1, by Gilbert Strang, et al.
- Community Calculus by David Guichard, et al.
- APEX Calculus by Gregory Hartman, et al.
- Paul's Online Calculus I Notes by Paul Dawkins
- Khan Academy Calculus I
- Prof. Shira Viel's remote instruction Math 111L videos (similar slides used in class)
- Prof. Sarah Schott's Calc II YouTube channel (has some Calc I review)
- Prof. Rann Bar-On's 105L materials (has precalc and Calc I content)

Learning objectives

The remainder of this coursepack is dedicated to practice with and reference on the course content learning objectives. These are listed fully below, and then organized to match the course sequence in the subsequent sections.

Prerequisite learning objectives

0. Work with **functions** and other precalculus mathematics proficiently.

    (a) Translate between function presentations, including graphs, formulas, and tables.
    (b) Recognize and work with elementary functions (i.e., linear, polynomial, rational, exponential, logarithmic, trigonometric, and inverse trig functions and their sums, differences, products, quotients, roots, compositions, and inverses), as well as piecewise-defined functions comprised of elementary functions (such as absolute value functions), including identifying domains and important graphical features.
    (c) Define and identify even and odd functions and their graphical properties.
    (d) Use appropriate algebraic techniques, terminology, and notation to solve equations, systems of two equations, and inequalities ('sign analysis'), including applying properties of exponents and logarithms.
    (e) State the definition of an invertible function, and determine whether or not a given function is invertible, including on restricted domain.
    (f) Utilize right triangle trigonometry and similar triangles.

Course learning objectives

1. Calculate, use, and explain the idea of **limits**.

    (a) Explain the concept of the limit of a function, both at a point (one-sided and two-sided) and as the input increases or decreases without bound.
    (b) Evaluate limits (or explain why a limit does not exist) using appropriate justification and notation, via both algebraic and graphical methods.
    (c) State and utilize the (limit) definition of continuity, including to determine where a function is continuous (or what conditions are required to ensure a function is continuous) using appropriate justification and notation, via both algebraic and graphical methods.
    (d) Translate between given information about the limits and values of a function and features of the graph of the function, including holes, vertical asymptotes, and horizontal asymptotes/end-behavior.

2. State and apply the definition and interpretations of the **derivative**.

    (a) Distinguish between speed & velocity; describe relationships with position and acceleration.
    (b) State, explain, and utilize the (limit) definitions of the derivative at a point and the derivative function.
    (c) Explain the connection between average and instantaneous rates of change, and interpret these concepts graphically using secant lines, tangent lines, and limits.

   (d) Find (equations of) the tangent line to a function at a point and the secant line between 2 points.
   (e) Use derivative notation correctly (both Lagrange $f'(x)$ and Leibniz dy/dx).
   (f) Compute higher-order derivatives (second, third, etc.).
   (g) Interpret the meaning of a derivative in context.
   (h) Identify points at which a function is (and is not) differentiable, using the definition of the derivative as justification.
   (i) Explain and utilize the relationship between differentiability and continuity of a function at a point.

3. Utilize **derivative rules** to calculate derivatives and explain the derivation of these rules from the definition of the derivative.

   (a) Explain the proofs of the constant-multiple rule, sum/difference-rule, and power rule for computing derivatives.
   (b) Compute derivatives of power, exponential, trigonometric, logarithmic, and inverse trigonometric functions.
   (c) Compute derivatives of constant multiples, sums, differences, products, quotients, compositions, piecewise-definitions (including absolute value), and inverses of functions.
   (d) Demonstrate why the derivative of an exponential function is proportional to the function itself.
   (e) Demonstrate how to derive the quotient rule using the product rule.
   (f) Prove the derivative rules for $\tan x, \cot x, \sec x$, and $\csc x$ (using the derivatives of $\sin x$ and $\cos x$).
   (g) Explain how to use the definition of inverse functions and the Chain Rule to find the derivative of $f^{-1}(x)$ given the derivative of $f(x)$, e.g., $\arctan x$.

4. Use derivatives to analyze functions and solve **applications**.

   (a) Recognize, explain, and interpret the relationships among the behaviors of f, f', and f'', including slopes, concavity, roots, critical points, inflection points, and local and global extrema.
   (b) Use the information provided by f, f', and/or f'' to identify and draw accurate graphs of the other functions.
   (c) Find critical points, local and global extrema and inflection points of functions both graphically and algebraically.
   (d) Use linearization to approximate function values with tangent lines, classifying as over- or underestimates via concavity.
   (e) State and apply the Mean Value, Rolle's, and Extreme Value Theorems.
   (f) Use implicit differentiation to perform logarithmic differentiation and to analyze implicitly defined curves.
   (g) Solve related rates and optimization word problems completely and correctly.
   (h) Identify limits of indeterminate form and use L'Hôpital's Rule to evaluate them, as well as to determine function dominance.

5. Explain the meaning and interpretations of definite and indefinite **integrals** and compute them and their approximations.

    (a) Approximate the value of a definite integral using left-, right-, and midpoint Riemann sums as well as trapezoidal sums and recognize when each is an overestimate, underestimate, or exact.

    (b) Write and recognize a definite integral as the limit of left-, right-, and midpoint Riemann sums as well as trapezoidal sums.

    (c) Compute the exact value of a definite integral using the Fundamental Theorem of Calculus and antiderivatives, geometry, and/or properties of definite integrals.

    (d) Find antiderivatives by using 'derivative rules backwards' coupled with guess-check-refine for compositions with linear functions like e^{2x} and $\frac{1}{3x+1}$.

    (e) Articulate the difference between the general antiderivative / indefinite integral of a function and a particular antiderivative, as well as an accumulation function.

    (f) Explain the meanings of the Fundamental Theorem of Calculus, Riemann Sums, and definite integrals in terms of a graph, and interpret them using the ideas of rate of change, net change, displacement, and the accumulation function.

    (g) Calculate the net change and average value of a function on an interval.

    (h) Compute the derivative of an accumulation function and compositions thereof.

6. Solve and approximate solutions to **differential equations** and utilize for modeling.

    (a) Articulate the definitions of and differences between general and particular solutions to DEs and initial value problems (IVPs), and verify whether or not a given function is a solution.

    (b) Identify separable DEs and solve using the method of separation of variables.

    (c) Use slope fields and Euler's method to approximate solutions to DEs and IVPs.

    (d) Find and classify equilibria of differential equations.

    (e) Write and interpret DEs/IVPs to mathematically model real processes, including motion, exponential growth and decay, and rate in/out.

Contents

0 Review of algebra and functions

Learning objectives (prerequisites)

0. Work with ***functions*** and other precalculus mathematics proficiently.

   (a) Translate between function presentations, including graphs, formulas, and tables.
   (b) Recognize and work with elementary functions (i.e., linear, polynomial, rational, exponential, logarithmic, *trigonometric, and inverse trig*[1] functions and their sums, differences, products, quotients, roots, compositions, and inverses), as well as piecewise-defined functions comprised of elementary functions (such as absolute value functions), including identifying domains and important graphical features.
   (c) Define and identify even and odd functions and their graphical properties.
   (d) Use appropriate algebraic techniques, terminology, and notation to solve equations, systems of two equations, and inequalities ('sign analysis'), including applying properties of exponents and logarithms.
   (e) State the definition of an invertible function, and determine whether or not a given function is invertible, including on restricted domain.
   (f) *Utilize right triangle trigonometry and similar triangles.*

Problem 1. For each of the 12 functions below,

- Sketch an approximate graph (by hand) and state the domain. (In our class, unless specified,the domain of a function is the largest collection of real numbers for which the function is defined (with a real output).)
- Identify any ***roots/zeroes***, $y-$intercept, and ***horizontal or vertical asymptotes***
- Determine if the function is ***even***, ***odd***, or neither. (A function f is even if $f(-x) = f(x)$ for all x in the domain, and odd if $f(-x) = -f(x)$ for all x in the domain.)

(a) $f_1(x) = x$	(d) $f_4(x) = x^3$	(g) $f_7(x) = \frac{1}{x-3}$	(j) $f_{10}(x) = \|x\| - 3$
(b) $f_2(x) = x - 4$	(e) $f_5(x) = \sqrt{x+4}$	(h) $f_8(x) = \|x\|$	(k) $f_{11}(x) = e^x$
(c) $f_3(x) = x^2 - 4$	(f) $f_6(x) = \frac{1}{x}$	(i) $f_9(x) = \|x - 3\|$	(l) $f_{12}(x) = \ln x$

Problem 2. Consider the following (piecewise) function:

$$f(x) = \begin{cases} \frac{1}{x} & x < -2 \\ 3 & x = -2 \\ x - 3 & -2 < x \leq 0 \\ \frac{1}{x-3} & x > 2 \end{cases}$$

(a) Compute each of the following values of the function f, if the value exists. If not, write "DNE" for "Does Not Exist."

[1]We'll review trigonometry later in the course, including sinusoidal functions and right triangle trig. There is no trig on the Calculus I Placement Test.

i. $f(-3)$ ii. $f(-2)$ iii. $f(0)$ iv. $f(1)$ v. $f(2)$ vi. $f(4)$

(b) Find the domain of $f(x)$. (Careful: you will need to consider each piece of the function)

Problem 3. Let $f(x) = x - 2$, $g(x) = \frac{x^2}{x-2}$, and $h(x) = x^2$.

(a) Is $f(x)$ invertible? If so, find $f^{-1}(x)$. If not, briefly explain why not.

(b) Is $h(x)$ invertible? If so, find $h^{-1}(x)$. If not, briefly explain why not.

(c) Find $(f \circ g)(x)$. (Recall $\circ$ denotes function composition: $(f \circ g)(x) = f(g(x))$.)

(d) Find $(g \circ f)(x)$.

(e) Consider the product function $m(x) = (g \cdot f)(x) = (f \cdot g)(x)$. (Recall $\cdot$ denotes multiplication: $(g \cdot f)(x) = (gf)(x) = f(x) \cdot g(x)$.)
Does $m(x) = h(x)$? Briefly justify your answer.

Problem 4. Fully simplify $\dfrac{\sqrt[3]{27x^5y^{12}}}{\sqrt[4]{x^3}}$ (so that x and y each occur once in your simplified expression). (From Prof. Rann Bar-On.)

Problem 5. Find and correct the mistakes in each of the following equation solutions. (From Prof. Sarah Schott.)

(a) $x^2 + 3x + 2 = 6 \implies (x+2)(x+1) = 6 \implies x = -2, x = -1$

(b) $x^2 + 1 = x + 1 \implies x^2 = x \implies x = 1$

(c) $(x+2)^2 = 4 \implies x^2 + 2^2 = 4 \implies x^2 = 0 \implies x = 0$

(d) $\frac{1}{1+x} = 2 \implies 1 + \frac{1}{x} = 2 \implies \frac{1}{x} = 1 \implies x = 1$

Problem 6. Solve each of the following 4 equations. (That is, find the exact value of all real number values of x for which the equation holds true.) (From Prof. Rann Bar-On.)

(a) $4^{1-x} = 3^{2x+5}$

(b) $x^2e^{2x} + 2xe^{2x} = 8e^{2x}$

(c) $\ln(x^2 - 2x - 2) = 0$

(d) $\frac{1}{x+2} + \frac{1}{x+3} = 1$

Problem 7. Solve each of the following 2 inequalities. (That is, find all real number values of x for which the equation holds true. Generally, the solution set of an inequality consists of one or more intervals.)

(a) $|2x - 1| \leq 6$

(b) $x^2 - 2x + 5 > 8$

Problem 8. Rewrite $f(x) = |x^3 - 2x^2 - 3x|$ as a piecewise function, with no absolute value signs and domain pieces written in terms of x.
(Recall $|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$: for example, $|5x - 3| = \begin{cases} 5x - 3 & x \geq \frac{3}{5} \\ -(5x-3) & x < \frac{3}{5} \end{cases}$)

Further practice and reference:

- Sec 1.1-1.4 of Hughes-Hallett Calculus
- Chpt 1 of OpenStax Calculus
- *Review of Algebra* from Stewart Calculus
- Review at Paul's Online Notes
- Khan Academy lesson on piecewise and absolute value functions

1 Limits and continuity

Learning objectives

1. Define, calculate, and use **limits**, the foundation of Calculus.
   (a) Explain the concept of the limit of a function, both at a point (one-sided and two-sided) and as the input increases or decreases without bound.
   (b) Evaluate limits (or explain why a limit does not exist) using appropriate justification and notation, via both algebraic and graphical methods.
   (c) State and utilize the (limit) definition of continuity, including to determine where a function is continuous (or what conditions are required to ensure a function is continuous) using appropriate justification and notation, via both algebraic and graphical methods.
   (d) Translate between given information about the limits and values of a function and features of the graph of the function, including holes, vertical asymptotes, and horizontal asymptotes/end-behavior.

Problem 1. Consider the following 4 functions:

$$a(x) = \begin{cases} \dfrac{x^4 - x^3}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases} \qquad b(x) = \begin{cases} x^3, & x < 1 \\ 2, & x \geq 1 \end{cases} \qquad c(x) = \frac{1}{x-1} \qquad d(x) = \frac{1}{|x-1|}$$

(*a*) Compute the limit as $x \to 1^-$, as $x \to 1^+$, and as $x \to 1$ of each function, being sure to show your work to justify your reasoning. (Graphical justification is fine if you can draw the graphs by hand.) If a limit does not exist and you can still say something definitive about the behavior of the function, use the appropriate notation for doing so.

(*b*) Determine whether or not each function is (i) continuous everywhere, and (ii) continuous on its domain, justifying your answers. (Note you will need to identify the domain!)

Problem 2. Determine whether each of the following statements is true or false. (Note that a statement is only true if it is always true, in all possible cases.) If true, just say so, but if false, justify why by either providing a brief explanation or giving a ***counterexample*** to the implication statement: a specific case that satisfies the ***hypotheses*** of the implication (the "If" part) but not the ***conclusion*** (the "Then" part).

(*a*) If $f(x)$ is continuous at $x = c$ then both $\lim_{x \to c} f(x)$ and $f(c)$ exist.

(*b*) If both $\lim_{x \to c} f(x)$ and $f(c)$ exist then $f(x)$ is continuous at $x = c$.

(*c*) If $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$ then $f(x)$ is continuous at $x = c$.

(*d*) If $f(x)$ is not continuous at $x = c$, then $\lim_{x \to c} f(x)$ does not exist.

Problem 3. For each set of criteria below, provide an example of a function that satisfies all of the given criteria, and briefly explain why using limits.

(*a*) Horizontal asymptote: $y = 1$, Vertical asymptote: none.

(*b*) Horizontal asymptote: none, Vertical asymptote: $x = -1$.

(*c*) Horizontal asymptote: $y = 1$, Vertical asymptote: $x = -1$.

Problem 4. Consider the function $f(x) = \begin{cases} x^2 + 4, & x \leq 1 \\ \sqrt{x + 15} - c, & x > 1. \end{cases}$.

Find all values, if any, of the real-number constant c that makes $f(x)$ continuous everywhere. (Be sure to clearly state and use the (limit) definition of continuity, use correct notation throughout, and justify why $f(x)$ is continuous everywhere, not just at $x = 1$.)

Problem 5. Compute the following limits from Paul's Online Notes, using our notation from class for functions which diverge to $\pm\infty$. (Feel free to do more for extra practice!)

(*a*) Sec 2.4: Problem 2

(*b*) Sec 2.5: Problems 6,9

(*c*) Sec 2.6: Problems 2,5

(*d*) Sec 2.7: Problems 3,8

(*e*) Sec 2.8: Problems 3,7,9

Further practice and reference:

- Sec 1.7-1.8 of Hughes-Hallett Calculus
- Sec 1.2 of Active Calculus
- Chpt 2 and Sec 4.6 of OpenStax Calculus Sec 2.2-2.9 of Paul's Online Math Notes

2 Definition of the derivative

2.1 Speed & velocity + Secant & tangent lines + the derivative

Learning objectives

2. Understand the meaning of the **derivative**.
   (a) Distinguish between speed & velocity; describe relationships with position.
   (b) State, explain, and utilize the (limit) definitions of the derivative at a point and the derivative function.
   (c) Explain the connection between average and instantaneous rates of change, and interpret these concepts graphically using secant lines, tangent lines, and limits.
   (d) Find (equations of) the tangent line to a function at a point and the secant line between 2 points.

Problem 1. A baseball is thrown straight up in the air from a height of 6 feet above ground, and the data below is collected, where time t is measured in seconds since the ball was thrown and $s(t)$ denotes the height of the baseball above the ground, in feet, at time t:

t (sec)	1	2	3	4	5	6
$s(t)$ (ft)	90	142	162	150	106	30

(a) What is $s(0)$?

(b) Find the average velocity of the baseball between 1 and 3 seconds after it was thrown and between 5 and 6 seconds after it was thrown.

(c) Find the average speed of the baseball between 1 and 3 seconds after it was thrown and between 5 and 6 seconds after it was thrown.

(f) Is the the average velocity between 1 and 3 seconds an over- or underestimate of the instantaneous velocity of the baseball 1 second after it was thrown? Briefly justify.

(g) Is the the average speed between 5 and 6 seconds an over- or underestimate of the instantaneous speed of the baseball 5 seconds after it was thrown? Briefly justify.

(h) Is the the average velocity between 5 and 6 seconds an over- or underestimate of the instantaneous velocity of the baseball 5 seconds after it was thrown? Briefly justify.

Problem 2. Which of the following limits, if any, gives the derivative of a function $f(x)$ at $x = 3$? Select all that apply.

○ $\lim_{b\to 3} \frac{f(b) - f(3)}{b - 3}$

○ $\lim_{x\to 3} \frac{f(x) - f(3)}{x - 3}$

○ $\lim_{x\to h} \frac{f(x) - f(h)}{3 - h}$

○ $\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$

○ $\lim_{h\to 0} \frac{f(3+h) - f(h)}{h}$

Problem 3. Consider the function $f(x) = x^2$.

(a) Find the slope-intercept form of the equation of the secant line to the graph of $f(x)$ between $x = 2$ and $x = 4$.

(b) Find the slope-intercept form of the equation of the tangent line to the graph of $f(x)$ at $x = 2$, using the limit definition of the derivative to find the slope.

Problem 4. Suppose that the tangent line to the graph of $y = f(x)$ at $x = 1$ is defined by the equation $y = 4x - 2$.

(a) Find $f(1)$.

(b) Find $f'(1)$.

Further practice and reference:

- Sec 2.1-2.2 of Hughes-Hallett Calculus
- Sec 1.3 of Active Calculus
- Sec 3.1 of OpenStax Calculus
- Definition of Derivative Practice Worksheet
- Section 3.1 Practice Problems of Paul's Online Notes

2.2 Derivative function + interpretation; Higher-order derivatives

Learning objectives

2. Understand the meaning of the **derivative**.
   (a) Distinguish between speed & velocity; describe relationships with position and acceleration.
   (b) State, explain, and utilize the (limit) definitions of the derivative at a point and the derivative function.
   (e) Use derivative notation correctly (both Lagrange $f'(x)$ and Leibniz dy/dx), including for higher-order derivatives.
   (f) Compute higher-order derivatives (second, third, etc.).
   (g) Interpret the meaning of a derivative in context.

4. Use derivatives to understand and solve **applications**.
   (a) Recognize, explain, and interpret the relationships among the behaviors of f, f', and f'', including slopes and concavity.
   (b) Use the information provided by f, f', and/or f'' to identify and draw accurate graphs of the other functions.

Problem 1. Determine the sign (positive, negative, or zero) of both the velocity and the acceleration function of a car in each of the following situations:

(a) The car is moving forward (in the positive direction) and speeding up

(b) The car is moving backward and maintaining a constant speed

(c) The car is moving backward (in the negative direction) and speeding up

Problem 2. Suppose a particle moves back and forth along the x-axis with position function $s(t) = 2t^3 - 4t$ where t is measured in seconds since you began observing and position is measured in micrometers (μm).

(a) Find the particle's velocity, acceleration, and jerk functions, using the limit definition. (***Jerk*** is the rate of change of acceleration.)

(b) What is the particle's acceleration 2 seconds after you began observing? (Include units!)

(c) During what period(s) of time is the particle slowing down? (Consider all time, not just time since you began observing.)

Problem 3. Suppose the volume of gasoline, in gallons, in a tank t minutes after 9am is given by the following function:

$$V(t) = \frac{2t^2}{t+1}$$

(a) Compute $V'(t)$ using the limit definition and explain its physical meaning in words, including units.

(b) Find any instant(s) in time at which the volume of gas in the tank is constant (recall that constant means "not changing").

Problem 4. Math 111L graphing derivatives practice

Further practice and reference:

- Sec 2.3-2.5 of Hughes-Hallett Calculus
- Sec 1.4-1.6 of Active Calculus
- Sec 3.2 and 3.4 of OpenStax Calculus
- Section 3.1-3.2 Practice Problems of Paul's Online Notes

2.3 Differentiability

Learning objectives

2. Understand the meaning of the **derivative**.

   (h) Identify points at which a function is (and is not) differentiable, using the definition of the derivative as justification.

   (i) Explain and utilize the relationship between differentiability and continuity of a function at a point.

Problem 1. Consider the function $f(x)$ below. Find $f'(1)$ or explain why it doesn't exist.

$$f(x) = \begin{cases} 2x+3, & x \leq 1 \\ 2x-5, & x > 1 \end{cases}$$

Problem 2. Consider the function $f(x)$ below. Find $f'(0)$ or explain why it doesn't exist. (You may use derivative results provided in class or the limit definition of the derivative.)

$$f(x) = (x + |x|)^2$$

Problem 3. Determine whether each of the following statements is true or false. (Note that a statement is only true if it is always true, in all possible cases.) If false, briefly justify by either an explanation or a counterexample.

(a) If a function $f(x)$ is continuous at $x = a$ then $f(x)$ is differentiable at $x = a$.

(b) If a function $f(x)$ is not differentiable at $x = a$ then $f(x)$ is not continuous at $x = a$.

(c) If a given function $f(x)$ is differentiable at $x = a$, then $f''(a)$ exists.

Problem 4. Consider the piecewise-defined function

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x^2, & \text{if } 0 < x \leq 3, \\ ax+b, & \text{if } x > 3. \end{cases}$$

(a) Is $f(x)$ differentiable at $x = 0$? If so, compute $f'(0)$.

(b) Find the values of real numbers a and b such that $f(x)$ is differentiable for all $x > 0$, being sure to fully and clearly justify your work so your reasoning is clear, and to use correct notation throughout. (You may use derivative results provided in class or the limit definition of the derivative.)

Further practice and reference:

- Sec 2.6 of Hughes-Hallett Calculus
- Sec 1.7 of Active Calculus
- Sec 3.2 OpenStax Calculus

3 Computing derivatives

3.1 Power rule + Sum, Difference, and Constant multiple rules

Learning objectives

3. Utilize **derivative rules** and explain their derivation.
   (a) Explain the proofs of the constant-multiple rule, sum/difference-rule, and power rule for computing derivatives.
   (b) Compute derivatives of power functions.
   (c) Compute derivatives of constant multiples, sums, differences, and piecewise-definitions (including absolute value) of functions.

Problem 1. This question is precalculus review both to help with rewriting power functions, and in preparation for next time, when we will be talking about exponential functions. Let a, b, c, m, and n be real number constants with $a, b > 0$ and x and y be real number variables. Which statements below are true? (Remember, a statement is only true if it is ALWAYS true, for any values of the variables. Otherwise, it is false.)

1. $\dfrac{a^{-m}}{b^{-n}} = \dfrac{b^n}{a^m}$
2. $x^m + x^n = x^{m+n}$
3. $2^{3x} = 8^x$
4. $x^m x^n = x^{mn}$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
6. $x^m x^n = x^{m+n}$
7. $\sqrt[m]{x} = x^{-m}$
8. $\sqrt[m]{x} = x^{1/m}$
9. $(ax)^n = a^n x^n$

Problem 2. Prove that $\dfrac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$

Problem 3. Examples 1-5 at Paul's Online Notes, Sec 3.3

Problem 4. Math 111L absolute value limits and derivatives practice

Further practice and reference:

- Sec 3.1 of Hughes-Hallett Calculus
- Sec 2.1 of Active Calculus
- Sec 3.3 of OpenStax Calculus
- Section 3.3 Practice Problems of Paul's Online Notes

3.2 Derivatives of exponential functions; Product & quotient rules

Learning objectives

3. Utilize **derivative rules** and explain their derivation.
   (b) Compute derivatives of power and exponential functions.
   (c) Compute derivatives of constant multiples, sums, differences, products, quotients, and piecewise-definitions (including absolute value) of functions.
   (d) Demonstrate why the derivative of an exponential function is proportional to the function itself.
   (e) Demonstrate how to derive the quotient rule using the product rule.

Problem 1. Use the product rule (twice) to prove that

$$\frac{d}{dx}\big(f(x)g(x)h(x)\big) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

(Hint: Consider the product of 3 functions as $f(x)g(x)h(x) = f(x)\cdot\big(g(x)h(x)\big)$).

Problem 2. Which of the following statements, if any, are true? Select all that apply. (Remember, a statement is only true if it is ALWAYS true: otherwise, it is false.)

- ○ Exponential functions are differentiable everywhere.
- ○ The rate of change of an exponential function is constant.
- ○ The rate of change of an exponential function is proportional to the function itself.
- ○ The derivative of an exponential function is an exponential function.
- ○ If $f(x)$ is an exponential function, then $f^{(n)}(x)$ is an exponential function for any natural number (positive integer) n.
- ○ The ratio between the rate of change of an exponential function at a point and the value of the function at that point is constant.

Problem 3. Which of the following statements, if any, are true? Select all that apply. (Remember, a statement is only true if it is ALWAYS true: otherwise, it is false.)

- ○ The derivative of a sum is the sum of the derivatives
- ○ The derivative of a difference is the difference of the derivatives
- ○ The derivative of a product is the product of the derivatives
- ○ The derivative of a quotient is the quotient of the derivatives
- ○ The rate of change of the area of a rectangle with side lengths $f(x)$ and $g(x)$ is $f'(x)g'(x)$
- ○ The rate of change of the perimeter of a rectangle with side lengths $f(x)$ and $g(x)$ is $f'(x) + g'(x)$
- ○ The rate of change of the area of a square with side length $f(x)$ is $2f(x)f'(x)$

Problem 4. Let $f(x) = x5^x$. Where is $f(x)$ concave down?

Problem 5. Examples 1-3 at Paul's Online Notes, Sec 3.4

Further practice and reference:

- Sec 3.2-3.3 of Hughes-Hallett Calculus
- Sec 2.1 and 2.3 of Active Calculus
- Sec 3.3 of OpenStax Calculus
- Sec 3.4 Practice Problems of Paul's Online Notes
- Sec 3.6 Practice Problems 1,5,7,9 of Paul's Online Notes

3.3 Chain rule

Learning objectives

3. Utilize **derivative rules** and explain their derivation.

   (c) Compute derivatives of constant multiples, sums, differences, products, quotients, compositions, and piecewise-definitions (including absolute value) of functions.

Problem 1. Which of the following statements, if any, are true? Select all that apply. (Remember, a statement is only true if it is ALWAYS true: otherwise, it is false.)

- ○ The notation $(f \circ g)(x)$ means "apply function g to x first, then apply function f to the result."
- ○ The notation $(f \circ g)(x)$ means "apply function f to x first, then apply function g to the result."
- ○ The notation $f(g(x))$ means "apply function f to x first, then apply function g to the result."
- ○ The notation $f(g(x))$ means "apply function g to x first, then apply function f to the result."
- ○ The rate of change of the area of a square of side length $s(x)$ is $2s(x)s'(x)$.
- ○ The rate of change of the area of a square of side length $s(x)$ is $2s'(x)$.
- ○ The rate of change of the perimeter of a square of side length $s(x)$ is $2s'(x)$.
- ○ The rate of change of the perimeter of a square of side length $s(x)$ is $4s'(x)$.

Problem 2. Compute each of the following given the functions below:

$f(x) = 2x^3+x-15, \quad g(x) = 2e^x-3x \quad h(x) = x^2+7 \quad m(x) = (f \circ g)(x) \quad n(x) = h(f(g(x)))$

(a) $g(0)$

(b) $g'(0)$

(c) $f(g(0))$

(d) $f'(g(0))$

(e) $m'(0)$ (Hint: Use your answers to previous parts!)

(f) $h'(f(g(0))$

(g) $n'(0)$ (Hint: Use your answers to previous parts!)

Problem 3. Examples 1, 2c, 3a, 3b, 4c, 4d, 5a, and 5b at Paul's Online Notes, Sec 3.9

Further practice and reference:

- Sec 3.4 of Hughes-Hallett Calculus
- Sec 2.5 of Active Calculus
- Sec 3.6 of OpenStax Calculus
- Sec 3.9 Practice Problems 1-3, 7, 14, 17, 28, 29, 31 of Paul's Online Notes

3.4 Review of trigonometry + Derivatives of trig functions

Learning objectives

0. Work with ***functions*** and other precalculus mathematics proficiently.
   (a) Translate between function presentations, including graphs, formulas, and tables.
   (b) Recognize and work with linear, polynomial, rational, absolute value, exponential, logarithmic, trigonometric, and *inverse trigonometric*[2] functions and combinations thereof, including identifying domains and important graphical features.
   (f) Utilize right triangle trigonometry and similar triangles.
3. Utilize **derivative rules** and explain their derivation.
   (b) Compute derivatives of power, exponential, and trigonometric functions.
   (e) Prove the derivative rules for $\tan x, \cot x, \sec x$, and $\csc x$ (using the derivatives of $\sin x$ and $\cos x$).

Problem 1. (*a*) Fill in the table below: for each angle θ in radians, specify the angle measure in degrees and find the exact value of $\cos\theta, \sin\theta$, and $\tan\theta$ (or write DNE). *(Note that on Test 2 and Test 3 in this class, you will be provided with a table with the values of* $\cos\theta$ *and* $\sin\theta$ *at* $\pi/6, \pi/4$*, and* $\pi/3$*. You should know how to fill in the rest of chart, and if needed put it on your notes sheet for the final.)*

θ (radians)	θ (degrees)	$\cos\theta$	$\sin\theta$	$\tan\theta$
0				
$\frac{\pi}{6}$		$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	
$\frac{\pi}{4}$		$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	
$\frac{\pi}{3}$		$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	
$\frac{\pi}{2}$				

(*b*) Use your work above and the symmetry of the unit circle to find the following:

i. $\cos\left(\frac{2\pi}{3}\right)$ ii. $\sin\left(\frac{5\pi}{4}\right)$ iii. $\tan\left(\frac{11\pi}{6}\right)$

(*c*) Use your work above and your knowledge of the periodicity of the sine, cosine, and tangent functions to find the following:

i. $\sin\left(\frac{51\pi}{2}\right)$ ii. $\cos\left(\frac{11\pi}{3}\right)$ iii. $\tan\left(\frac{17\pi}{4}\right)$

[2] Covered in the next section

Problem 2. Which of the following functions, if any, are equivalent to the function $\cos x$? Select all that apply.

- ◯ $\cos(x + 4\pi)$
- ◯ $\sin(x + \frac{\pi}{2})$
- ◯ $\cos(x + 3\pi)$
- ◯ $\sin(x - \frac{\pi}{2})$
- ◯ $-\sin(x - \frac{\pi}{2})$
- ◯ $\cos(-x)$
- ◯ $\sin x \cot x$
- ◯ $\cos^3 x + \cos x \sin^2 x$

Problem 3. Questions 1-4 at Khan Academy, Modeling with sinusoidal functions.

Problem 4. Example 2 at Paul's Online Notes, Sec 3.5.

Further practice and reference:

- Sec 1.5 (Trig Review) and Sec 3.5 (Trig Derivs) of Hughes-Hallett Calculus
- Sec 2.2 and Sec 2.4 of Active Calculus
- Sec 1.3 (Trig Review) and Sec 3.5 (Trig Derivs) of OpenStax Calculus
- Sec 3.5 Practice Problems 4-14 of Paul's Online Notes

3.5 Review of inverse functions + Derivatives of inverse functions

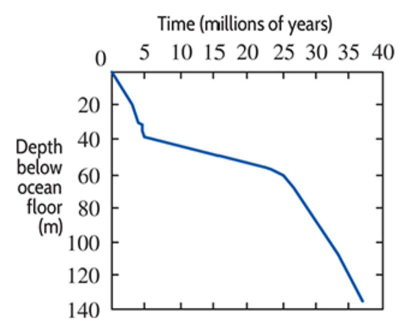
Learning objectives

0. Work with ***functions*** and other precalculus mathematics proficiently.
   (a) Translate between function presentations, including graphs, formulas, and tables.
   (b) Recognize and work with linear, polynomial, rational, absolute value, exponential, logarithmic, trigonometric, and inverse trigonometric functions and combinations thereof, including identifying domains and important graphical features.
   (d) Use appropriate algebraic techniques, terminology, and notation to help solve problems, including solving equations, systems of equations, and inequalities ('sign analysis'), and applying properties of exponents and logarithms.
   (e) State the definition of an invertible function, and determine whether or not a given function is invertible, including on restricted domain.

3. Utilize **derivative rules** and explain their derivation.
   (b) Compute derivatives of power, exponential, trigonometric, logarithmic, and inverse trigonometric functions.
   (c) Compute derivatives of constant multiples, sums, differences, products, quotients, compositions, piecewise-definitions (including absolute value), and inverses of functions.
   (f) Explain how to use the definition of inverse functions and the Chain Rule to find the derivative of $f^{-1}(x)$ given the derivative of $f(x)$, e.g., $\arctan x$.

Problem 1. The figure at right is a graph of the function $D = f(t)$, the depth in meters below the Atlantic Ocean floor where t million-year-old rock can be found.[a]

[a] Adapted from Hughes-Hallett *Calculus*. Data of Dr. Murlene Clark based on core samples drilled by research ship *Glomar Challenger*, from *Initial Reports of the Deep Sea Drilling Project*.



(a) Find the approximate value of $f(15)$, and describe its practical meaning in the context of the problem. Include units.

(b) Briefly describe why f invertible (on the depicted domain).

(c) Find the approximate value of $f^{-1}(120)$, and describe its practical meaning in the context of the problem. Include units.

(d) We see that f is an increasing function. (The graph makes it seem like it is decreasing, but be careful, the depth axis is reversed. As the age of rock increases, the depth at which it can be found increases as well.) Is f^{-1} increasing, decreasing, or neither? Briefly explain.

Problem 2. Find (the exact value of) each of the following derivatives.

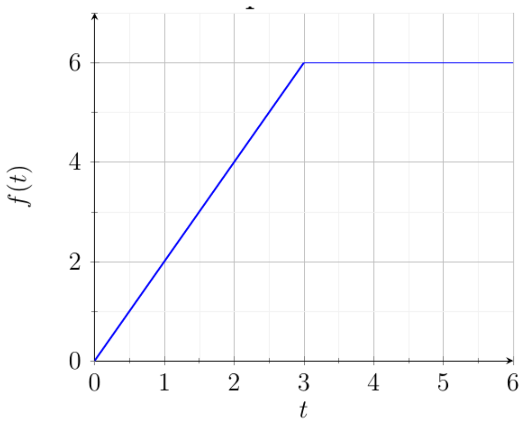
(a) $f'(1)$ if $f(x) = 7\ln(2x^2 + 5)$.

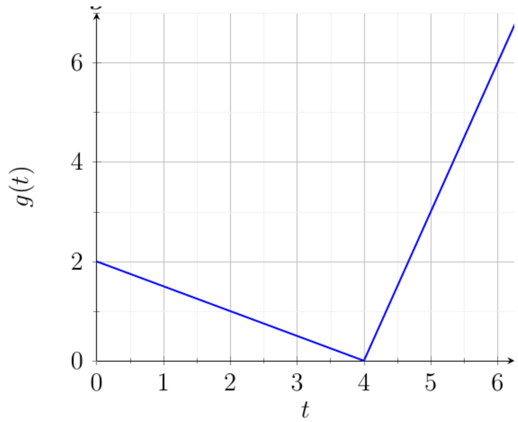
(b) $\dfrac{d}{dx}\left(4\arcsin(3x)\right)\Big|_{x=1/5}$

Problem 3. Suppose f and g are both differentiable, invertible functions with differentiable inverses f^{-1} and g^{-1}, respectively, and define the function $h(t) = (f \circ g)(t)$.

(a) Express $h^{-1}(t)$, the inverse function of h, in terms of f and g. (Hint: Remember, we require that $h^{-1}(h(t)) = t$. To "undo" $h(t) = f(g(t))$, which function do you need to undo first? f or g? Which function do you need to undo next?)

(b) Consider the functions $f(t)$ and $g(t)$ whose graphs are depicted below.





i. Assuming the graphs continue as shown, give the formulas for $f(t)$ and $g(t)$.

ii. Describe, algebraically, why neither $f(t)$ nor $g(t)$ is a one-to-one function.

iii. Give a subinterval of $[0, 6]$ such that both $f(t)$ and $g(t)$ are invertible when restricted to that domain.

iv. Use the graphs to find the exact value of each of the following quantities. (Recall that $h(t) = (f \circ g)(t)$.)

- $g(2)$
- $h(2)$
- $f^{-1}(2)$
- $h^{-1}(2)$
- $g'(2)$
- $f'(g(2))$
- $h'(2)$
- $(h^{-1})'(2)$

Further practice and reference:

- Sec 1.3 (Inverse Function Review) and Sec 3.5 (Derivatives of Inverse functions) of Hughes-Hallett Calculus
- Sec 2.6 of Active Calculus
- Sec 1.4 (Inverse Function Review) and Sec 3.7 (Derivatives of Inverse functions) of OpenStax Calculus
- Sec 3.6 Practice Problems 1-10 and Sec 3.7 Practice Problems 4-5 of Paul's Online Notes

4 Applications of the derivative

Learning objectives

4. Use derivatives to understand and solve **applications**.

   (d) Use ***linearization*** to approximate function values with tangent lines, classifying as over- or underestimates via concavity.

4.1 Linear approximation

Problem 1. Let $f(x)$ be a function, a a real number at which $f(x)$ is differentiable, and $L(x)$ the linearization of $f(x)$ at $x = a$. Which of the following statements, if any, are true? Select all that apply. (Remember, a statement is only true if it is ALWAYS true: otherwise, it is false.)

- ◯ If $f''(x) > 0$ on an interval around $x = a$, then $L(x) < f(x)$ for values of $x \neq a$ in the domain of f in that interval.
- ◯ If $f''(x) > 0$ on an interval around $x = a$, then $L(x) > f(x)$ for values of $x \neq a$ in the domain of f in that interval.
- ◯ If $f''(x) > 0$ on an interval around $x = a$, then $L(x) < f(x)$ for ALL values of $x \neq a$ in the domain of f.
- ◯ If $f''(x) < 0$ on an interval around $x = a$, then $L(x) < f(x)$ for values of $x \neq a$ in the domain of f in that interval.
- ◯ If $f''(x) < 0$ on an interval around $x = a$, then $L(x) > f(x)$ for values of $x \neq a$ in the domain of f in that interval.
- ◯ If $f''(x) < 0$ on an interval around $x = a$, then $L(x) > f(x)$ for ALL values of $x \neq a$ in the domain of f.

Problem 2. Suppose $f(x)$ is a ***twice-differentiable*** function defined everywhere ("twice-differentiable" means both $f'(x)$ and $f''(x)$ are defined everywhere), where $f'(x)$ is decreasing for all x. In each of the following pairs, which quantity is larger? Explain your reasoning. *(Hint: Think about slopes and linear approximations!)*

(a) $f'(3)$ and $f'(4)$

(b) $f''(3)$ and 0

(c) $f(3 + \Delta x)$ and $f(3) + f'(3)\Delta x$, where Δx is a small positive number.

Problem 3. Consider the following quantity:

$$(2.001)^4$$

Use linearization to estimate this value (by hand), and state whether your estimation is an underestimate, overestimate, or exact value. As always, clearly show each step of your work/justify your reasoning.
(Hint: think of this as a function whose value is being approximated near *a point where the value can be computed by hand: what is the function? What is the point?)*

Problem 4. Sec 4.11 Practice Problems 1-3 of Paul's Online Notes. Please also determine whether each linear approximation is an over- or underestimate of the function near the point of linearization, briefly justifying your reasoning.

Further practice and reference:

- Sec 3.9 of Hughes-Hallett Calculus
- Sec 1.8 of Active Calculus
- Sec 4.2 of OpenStax Calculus
- Sec 4.11 of Paul's Online Notes

4.2 Implicit differentiation: implicit curves, logarithmic diff, and related rates

Learning objectives

4. Use derivatives to understand and solve **applications**.

   (f) Use implicit differentiation to perform logarithmic differentiation and to analyze implicitly defined curves.

   (g) Solve related rates word problems completely and correctly.

Problem 1. Given the following implicitly-defined curve, find dy/dx:

$$x^3 + y^3 - xy^2 = 5$$

Problem 2. Sec 3.13 Examples 1-3 of Paul's Online Notes

Problem 3. Suppose that a snowball melts (i.e., its volume decreases) at a rate proportional to its current surface area. (Recall that two quantities A and B are ***(directly) proportional*** if there exists a ***constant of proportionality*** k such that $A = kB$.)

(a) Show that the radius of the ball decreases at a constant rate. (Recall that the surface area of a sphere of radius r is given by $4\pi r^2$, and the volume by $\frac{4}{3}\pi r^3$.)

(b) *CHALLENGE QUESTION* If the snowball melts to $\frac{1}{4}$ of its original volume in 10 minutes, how long will it take for the ball to melt entirely?

(*Hint: Let $r(0)$ denote the initial radius of the snowball, at time $t = 0$. In terms of $r(0)$, what is $r(10)$, when the snowball is $\frac{1}{4}$ of its initial volume? How can we use that information to find the constant rate of change of the snowball's radius? Finally, how can we use that rate of change to find out how long it takes for the radius to become 0?*)

Problem 4. An observer with a telescope watches a rocket travel straight upward from a launch pad located 10 km away. At a certain moment the angle between the telescope and the ground is $\pi/3$ and is changing at a rate of 0.5 rad/min. What is the rocket's velocity (in km/min) at that moment? (Question adapted from Active Calculus Activity 3.5.3.)

Problem 5. More Related Rates: Sec 3.11 Examples 1,2,3,5,7 of Paul's Online Notes

Further practice and reference:

- Sec 3.7 (Implicit Differentiation) and 4.6 (Related Rates) of Hughes-Hallett Calculus
- Sec 2.7 (Implicit Differentiation) and Sec 3.5 (Related Rates) of Active Calculus
- Sec 3.8 (Implicit Differentiation) and Sec 4.1 (Related Rates) of OpenStax Calculus
- Sec 3.10 (Implicit Differentiation), Sec 3.11 (Related Rates), and Sec 3.13 (Logarithmic Differentiation) Practice Problems of Paul's Online Notes

4.3 Using f' and f'': roots, extrema, IPs, and MVT

Learning objectives

4. Use derivatives to understand and solve **applications**.

   (a) Recognize, explain, and interpret the relationships among the behaviors of f, f', and f'', including slopes, concavity, roots, critical points, inflection points, and local extrema.

   (b) Use the information provided by f, f', and/or f'' to identify and draw accurate graphs of the other functions.

   (c) Find critical points, local extrema, and inflection points of functions both graphically and algebraically.

   (e) State and apply the Mean Value Theorem and Rolle's Theorem.

Problem 1. Which of the following statements, if any, are true? Select all that apply. (Remember, a statement is only true if it is ALWAYS true: otherwise, it is false.) For extra practice, for each statement which is false, briefly justify why, either with an explanation or a counterexample.

- ◯ If $f'(x) < 0$ on an interval, then f is decreasing on that interval.
- ◯ If $f'(x) > 0$ on an interval, then f is concave up on that interval.
- ◯ If $f''(x) < 0$ on an interval, then f is concave down on that interval.
- ◯ If $f''(x) > 0$ on an interval, then f is increasing on that interval
- ◯ If $f''(x) > 0$ on an interval, then f' is increasing on that interval

Problem 2. Which of the following statements, if any, are true? Select all that apply. (Remember, a statement is only true if it is ALWAYS true: otherwise, it is false.) For extra practice, for each statement which is false, briefly justify why, either with an explanation or a counterexample.

- ◯ If $f''(p) = 0$, then f has an inflection point at p.
- ◯ If $f''(x) < 0$ for all $a < x < p$ and $f''(x) > 0$ for $p < x < b$, and f is defined at $x = p$, then f has an inflection point at $x = p$.
- ◯ If $f'(p) = 0$ then f has a local extremum at $x = p$.
- ◯ If f has a critical point at $x = p$ then f has a local extremum at $x = p$.
- ◯ If f has a local extremum at p then $f'(p) = 0$.
- ◯ If f has a local extremum at $x = p$ then f has a critical point at $x = p$.
- ◯ None of the other statements are true.

Problem 3. Which of the following statements, if any, are true? Select all that apply. (Remember, a statement is only true if it is ALWAYS true: otherwise, it is false.) For extra practice, for each statement which is false, briefly justify why, either with an explanation or a counterexample.

- ◯ If $f(a) = f(b) = 0$ and f is differentiable on $[a, b]$, then $f'(c) = 0$ for some $a < c < b$.
- ◯ If $f(a) = f(b) = 0$ and f is continuous on $[a, b]$, then $f'(c) = 0$ for some $a < c < b$.
- ◯ If f is differentiable on $[a, b]$, then $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some $a < c < b$
- ◯ If f is differentiable on $[a, b]$, then $f'(c) = \frac{f'(b)-f'(a)}{b-a}$ for some $a < c < b$
- ◯ If f is a differentiable function, then there is at least one root of f' in between every two distinct roots of f.
- ◯ If f is a differentiable function, then there is at least one root of f in between every two distinct roots of f'.

Problem 4. Consider $f(x) = (x^2 - 4)^7$.

(a) Find the critical points of f.

(b) Find and factor $f''(x)$.

(c) Classify the critical points of f (i.e., find the locations of any local extrema). (Use whatever derivative test you wish that applies (recall the First Derivative Test always applies, but the Second Derivative Test might be quicker.))

(d) Find the inflection points, if any. (Leave in exact form.)

Problem 5. Does the function

$$f(x) = x^{999} + x^{99} + x^9 + x + 99$$

have any local extrema? Find their locations if so, and if not, explain if not.

Further practice and reference:

- Sec 3.10 and 4.1 of Hughes-Hallett Calculus
- Sec 3.1-3.2 of Active Calculus
- Sec 4.3-4.5 of OpenStax Calculus
- Sec 4.2-4.7 of Paul's Online Notes

4.4 Optimization

Learning objectives

4. Use derivatives to understand and solve **applications**.
   (b) Recognize, explain, and interpret the relationships among the behaviors of f, f', and f'', including slopes, concavity, roots, critical points, inflection points, and local and global extrema.
   (c) Find critical points, local and global extrema and inflection points of functions both graphically and algebraically.
   (e) State and apply the Mean Value, Rolle's, and Extreme Value Theorems.
   (g) Solve related rates and optimization word problems completely and correctly.

Problem 1. If possible, find both the values and the locations of the global maximum and global minimum of the function $f(x) = 2x^3 - 3x^2 - 12x + 5$ on each of the following 3 intervals. If either global extremum does not exist, explain why not. If an extremum does exist, justify why your given x-value(s) truly are location(s) where the extremum is achieved.

(a) $[-2, 3]$

(b) $(2, 3]$

(c) $(-2, 3)$

Problem 2. Which of the following statements, if any, are true? Select all that apply. (Remember, a statement is only true if it is ALWAYS true: otherwise, it is false.) For extra practice, for each statement which is false, briefly justify why, either with an explanation or a counterexample.

- ◯ Every function has both a global maximum and a global minimum on any set of real numbers.
- ◯ If a function has a local maximum (and/or minimum) on a given interval, than it must have a global maximum (and/or minimum) on that interval.
- ◯ Every function has both a global maximum and a global minimum on every closed interval on which it is defined.
- ◯ Every function has both a global maximum and a global minimum on every closed interval on which it is continuous.
- ◯ It is impossible for a function to achieve its global maximum on an interval at two different locations in that interval.
- ◯ It is impossible for a function to achieve two different global maximums on an interval.
- ◯ Every constant function has both a global maximum and a global minimum on every set of real numbers.

Problem 3. You are building a rectangular box with square base and open top to hold a volume of 8 cubic inches and want to build it as cheaply as possible. The material for the bottom of the box costs 6 cents per square inch, and the material for the sides costs 3 cents per square inch. What choice of side length s of the base of the box and height h minimizes the total cost of materials?

Problem 4. Sec 4.8 practice problems 1-8 of Paul's Online Notes. (Don't forget to clearly specify what (single-variable) function you are optimizing on what domain, and to justify that you have found the location of a *global* extremum (or, explain why none exists).)

Further practice and reference:

- Sec 4.2, 4.3, and 4.5 of Hughes-Hallett Calculus
- Sec 3.3-3.4 of Active Calculus
- Sec 4.7 of OpenStax Calculus
- Sec 4.8-4.9 of Paul's Online Notes

4.5 L'Hôpital's Rule

Learning objectives

4. Use derivatives to understand and solve **applications**.

   (h) Identify limits of indeterminate form and use L'Hôpital's Rule to evaluate them, as well as to determine function dominance.

Problem 1. Which of the following limits are of indeterminate form? Select all that apply. For extra practice, evaluate each limit, not just those which are IF. (Check your answers using a calculator or computational software. e.g., using `https://www.wolframalpha.com/`, one would type, for the first limit, `limit of (e^(2x)-1)/x as x approaches 0`. Also, recall you get free access to WolframAlpha Pro as Duke students, which will give you step-by-step solutions: go to `https://software.duke.edu/node/368`.)

- ◯ $\lim_{x\to 0} \frac{e^{2x}-1}{x}$
- ◯ $\lim_{x\to 0} \frac{\sin x}{x}$
- ◯ $\lim_{x\to 0} \frac{\cos x}{x}$
- ◯ $\lim_{x\to 0} \frac{e^{2x}}{x}$
- ◯ $\lim_{x\to \infty} \frac{3x+e^{-x}}{4x}$
- ◯ $\lim_{x\to -\infty} \frac{e^{x}}{7x}$
- ◯ $\lim_{x\to 0^+} \frac{\ln x}{x}$
- ◯ $\lim_{x\to 0^+} \frac{\ln x}{\cot x}$
- ◯ $\lim_{x\to 0^+} \frac{\sin x}{\cot x}$

Problem 2. Which of the following limits are of indeterminate form? Select all that apply. For extra practice, evaluate each limit. (Check your answers using computational software.)

- ◯ $\lim_{x\to 0^-} \frac{1}{x} - \frac{1}{\sin x}$
- ◯ $\lim_{x\to 0^-} \frac{1}{x} - \frac{1}{\cos x}$
- ◯ $\lim_{x\to 0^+} x\ln x$
- ◯ $\lim_{x\to 0} xe^{-x}$
- ◯ $\lim_{x\to \infty} xe^{-x}$
- ◯ $\lim_{x\to 0} xe^{-x}$

- ◯ $\lim_{x \to 0} (1+x)^x$
- ◯ $\lim_{x \to 0^+} x^x$

Problem 3. Sec 4.10 practice problems 1-5,7-11 of Paul's Online Notes.

Problem 4. Suppose that $f(x)$ is a differentiable function with continuous derivative $f'(x)$, where $f(4) = -2$, $f(6) = 2$, $f'(4) = -1$, $f'(6) = 10$. Evaluate the following:

$$\lim_{x \to 0} \frac{f(3x+4) + f(4x+6)}{2x}.$$

Problem 5. Compute the following limit:

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right)^x.$$

Further practice and reference:

- Sec 4.7 of Hughes-Hallett Calculus
- Sec 2.8 of Active Calculus
- Sec 4.8 of OpenStax Calculus
- Sec 4.10 of Paul's Online Notes

5 Integrals, antiderivatives, and the FTC

5.1 Definition, approximation, and interpretation of the definite integral

Learning objectives

5. Explain the meaning and interpretations of definite and indefinite **integrals** and compute them and their approximations.

   (a) Approximate the value of a definite integral using left-, right-, and midpoint Riemann sums as well as trapezoidal sums and recognize when each is an overestimate, underestimate, or exact.

   (b) Write and recognize a definite integral as the limit of left-, right-, and midpoint Riemann sums as well as trapezoidal sums.

   (c) Compute the exact value of a definite integral using geometry.

   (f) Explain the meanings of Riemann Sums and definite integrals in terms of a graph, and interpret them using the idea of rates of change, net change, and displacement.

Problem 1. Suppose that the velocity of an object at time t minutes after 1pm is given by $v(t) = 3t - 6$ meters/second.

(a) When is the object moving forwards?

(b) When is the object slowing down?

(c) Sketch a graph of the velocity function and use your graph to find the object's displacement on each of the following 3 intervals of time:

i. Between 1:00pm and 1:02pm

ii. Between 12:58pm and 1:01pm

iii. Between 1:01pm and 1:05pm

Problem 2. Find the exact value of each of the following 2 sums.

(a) $\displaystyle\sum_{i=1}^{3}(i^2 + i)$

(b) $\displaystyle\sum_{k=0}^{2}\frac{2}{3}\left(2 + \frac{2k}{3}\right)$

Problem 3. Sec 5.5 practice problems 1-3 of Paul's Online Notes.

Problem 4. Using sigma notation, write down each of the following 12 sums for the function $f(x) = \sqrt{3x^2 + 1}$ on the interval $2 \leq x \leq 7$ (where n is a positive integer).

$$LHS(10), RHS(10), MPS(10), TR(10)$$

$$LHS(100), RHS(100), MPS(100), TR(100)$$

$$LHS(n), RHS(n), MPS(n), TR(n)$$

Problem 5. Suppose $f(x)$ is a function where $f'(x) > 0$ and $f''(x) < 0$ for all $a \leq x \leq b$, and let n be a positive integer. Determine if each of the following sums for $f(x)$ on the interval $a \leq x \leq b$ is an overestimate or underestimate of the definite integral $\int_a^b f(x)\,dx$ or if it cannot be determined, justifying your answers.

(a) $RHS(n)$

(b) $LHS(n)$

(c) $MPS(n)$

(d) $TR(n)$

Problem 6. Sec 5.6 practice problems 8-9 of Paul's Online Notes.

Problem 7. (Adapted from Active Calculus Activity 4.4.4) A rower does a 40-minute workout on an erg (rowing machine). The rate $c(t)$ at which they burn calories, in kCal/min, is given by the following continuous function, where t is measured in minutes since they began rowing:

$$c(t) = \begin{cases} -0.05t^2 + t + 10, & 0 \leq t < 10 \\ 15, & 10 \leq t \leq 30 \\ -0.05t^2 + 3t - 30, & 30 < t \leq 40 \end{cases}$$

(a) Show that $c(t)$ is continuous on the time interval $0 \leq t \leq 40$. (This type of question won't be asked on Test 3, but is fair game for the final!)

(b) Show that $c(t)$ is differentiable on the time interval $0 \leq t \leq 40$. (This type of question won't be asked on Test 3, but is fair game for the final!)

(c) Compute the exact value of $\displaystyle\int_{10}^{30} c(t)\,dt$ and interpret it in the context of the problem.

(d) Suppose $C(t)$ is a function with $C'(t) = c(t)$. Approximate $C(40) - C(0)$ using a right-hand Riemann sum with 4 subintervals.

Further practice and reference:

- Sec 5.1-5.2 of Hughes-Hallett Calculus
- Sec 4.1-4.3 of Active Calculus
- Sec 5.1-5.2 of OpenStax Calculus
- Sec 5.5-5.6 of Paul's Online Notes

5.2 Antiderivatives & FTC I

Learning objectives

5. Explain the meaning and interpretations of definite and indefinite **integrals** and compute them and their approximations.
    (c) Compute the exact value of a definite integral using the Fundamental Theorem of Calculus and antiderivatives.
    (d) Find antiderivatives by using 'derivative rules backwards' coupled with guess-check-refine for compositions with linear functions like e^{2x} and $\frac{1}{3x+1}$.
    (e) Articulate the difference between the general antiderivative / indefinite integral of a function and a particular antiderivative.
    (f) Explain the meanings of the Fundamental Theorem of Calculus, Riemann Sums and definite integrals in terms of a graph, and interpret them using the idea of rates of change, net change, and displacement.
    (g) Calculate the net change of a function on an interval.

Problem 1. Suppose that ice cream is being scooped into a giant waffle cone, while also leaking out a hole in the bottom. The rate of change in the volume of ice cream in the cone, in cups per minute, t minutes after scooping begins, is given by

$$f(t) = -3t^2 + 7$$

(a) Compute $f(1)$, including units, and interpret it in context.

(b) Compute $\displaystyle\int_1^2 f(t)\,dt$, including units, and interpret it in context.

Problem 2. Suppose that $f(x) = 8x^3 + \sin(2\pi x) + \dfrac{1}{x+1} + 5$

(a) Find the general antiderivative of $f(x)$.

(b) Evaluate $\displaystyle\int_0^2 f(x)\,dx$ (leave in exact form).

(c) Find $F(x)$ if $F'(x) = f(x)$ and $F(0) = 3$.

Problem 3. Sec 5.2 practice problems of Paul's Online Notes.

Problem 4. Sec 5.7 practice problems 1-14 of Paul's Online Notes.

Further practice and reference:

- Sec 5.2-5.3 and 6.1-6.2 of Hughes-Hallett Calculus
- Sec 4.3-4.4 and Sec 5.1 of Active Calculus
- Sec 5.2-5.4 of OpenStax Calculus
- Sec 5.1-5.2 and Sec 5.6-5.7 of Paul's Online Notes

5.3 Properties of Definite Integrals

Learning objectives

5. Explain the meaning and interpretations of definite and indefinite **integrals** and compute them and their approximations.
   (c) Compute the exact value of a definite integral using the Fundamental Theorem of Calculus and antiderivatives, geometry, and/or properties of integrals.
   (g) Calculate the net change and average value of a function on an interval.

Problem 1. Suppose that $f(x)$ is an even function and $g(x)$ is an odd function, where

$$\int_{-2}^{2} f(x)\ dx = 8, \qquad \int_{2}^{4} f(x)\ dx = -3, \qquad \int_{2}^{8} f(x)\ dx = 10, \qquad \int_{-2}^{0} g(x)\ dx = 5$$

Compute each of the following, being sure to show each step of your work and explicitly state if using the fact that a given function is even or odd:

(a) $\displaystyle\int_{0}^{2} f(x)\ dx$

(b) $\displaystyle\int_{0}^{2} g(x)\ dx$

(c) $\displaystyle\int_{4}^{2} f(x)\ dx$

(d) $\displaystyle\int_{4}^{8} f(x)\ dx$

(e) $\displaystyle\int_{0}^{2} \big(f(x) + g(x) + 6\big)\ dx$

(f) The average value of $8f(x)$ on $2 \leq x \leq 4$

Problem 2. Sec 5.6 practice problems 4-7 of Paul's Online Notes.

Problem 3. Sec 5.7 practice problems 15-18 of Paul's Online Notes.

Further practice and reference:

- Sec 5.4 of Hughes-Hallett Calculus
- Sec 4.3-4.4 of Active Calculus
- Sec 5.2 of OpenStax Calculus
- Sec 5.1-5.2 and Sec 5.6-5.7 of Paul's Online Notes

5.4 Accumulation Function and FTC II

Learning objectives

5. Explain the meaning and interpretations of definite and indefinite **integrals** and compute them and their approximations.
   (e) Articulate the difference between the general antiderivative / indefinite integral of a function and a particular antiderivative.
   (f) Explain the meaning of the Fundamental Theorem of Calculus, Riemann Sums, and definite integrals in terms of a graph, and interpret and apply them using the ideas of rate of change, net change, displacement, and the accumulation function.
   (g) Articulate the difference between the general antiderivative / indefinite integral of a function and a particular antiderivative, as well as an accumulation function.
   (h) Compute the derivative of an accumulation function and compositions thereof.

Problem 1. Find an antiderivative $F(x)$ of $f(x) = \frac{e^x}{x}$ satisfying $F(3) = -1$.

Problem 2. Compute $\frac{d}{dx}\left(\int_{2x}^{\sin x} e^{t^3}\, dt\right)$.

Problem 3. Where is the following function concave up?

$$f(x) = \int_0^x \frac{t}{(1+t)^2} dt$$

Problem 4. Sec 5.6 practice problems 10-12 of Paul's Online Notes.

Further practice and reference:

- Sec 6.2 and 6.4 of Hughes-Hallett Calculus
- Sec 5.2 of Active Calculus
- Sec 5.3 of OpenStax Calculus
- Sec 5.1-5.2 and Sec 5.6 of Paul's Online Notes

6 Differential equations

6.1 Solutions to DEs and IVPs and Modeling Motion

Learning objectives

6. Solve and approximate solutions to **differential equations** and utilize for modeling.
   (a) Articulate the definitions of and differences between general and particular solutions to DEs and initial value problems (IVPs), and verify whether or not a given function is a solution.
   (e) Write and interpret DEs/IVPs to mathematically model real processes, including motion.

Problem 1. Consider the following IVP:

$$\frac{dy}{dx} = x^2 + 2y, \quad y(0) = 1$$

Show that $y = 3e^{2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}$ is a solution of the DE, but *not* a solution of the IVP.

Problem 2. Find the general solution of each of the following differential equations:

(a) $\dfrac{dy}{dx} = e^{3x} + 4$

(b) $\dfrac{dy}{dx} = \dfrac{2}{\cos^2(x/4)} + \dfrac{1}{2x+3}$

Problem 3. Suppose a golf ball hit off the roof of a 40-foot-tall building has velocity function $v(t) = -32t + 60$ feet/sec, where t is measured in seconds since the ball was hit and positive velocity corresponds to upwards motion.

(a) What is the ball's velocity 3 seconds after being hit? (Don't forget units.)

(b) What is the ball's speed 3 seconds after being hit? (Don't forget units.)

(c) Write and solve an IVP whose solution is $s(t)$, the position of the ball above the ground, in feet, t seconds after being hit.

(d) What is the ball's position 3 seconds after being hit?

Further practice and reference:

- Sec 6.3 and 11.1 of Hughes-Hallett Calculus
- Sec 7.1 of Active Calculus
- Sec 4.1 of OpenStax Calculus, vol. 2

6.2 Separation of Variables

Learning objectives

6. Solve and approximate solutions to **differential equations** and utilize for modeling.

    (b) Identify separable DEs and solve using the method of separation of variables.

    (d) Find constant solutions (equilibria) of differential equations.

Problem 1. For each DE below, (1) Find any constant solutions, and (2) Determine if the DE is separable, and if so, separate the variables as $\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$.

(a) $\frac{dy}{dx} = \frac{x}{y}$

(b) $\frac{dy}{dx} = -3y$

(c) $\frac{dy}{dx} = x + y$

(d) $\frac{dy}{dx} = \ln(xy)$

(e) $\frac{dy}{dx} = e^{x+y}$

(f) $\frac{dy}{dx} = 5x$

(g) $\frac{dy}{dx} = 2x - y$

(h) $\frac{dy}{dx} = 2y$

(i) $\frac{dy}{dx} = \ln(x^y)$

(j) $\frac{dy}{dx} = e^{xy}$

(k) $\frac{dy}{dx} = 2x - 2xy$

Problem 2. Consider the DE $\dfrac{dy}{dx} = 0.2y - 1$.

(a) Find a solution to the IVP $\dfrac{dy}{dx} = 0.2y - 1, \quad y(0) = 5$

(b) Solve the DE, being sure to specify the values of any arbitrary constants in your answer.

(c) Solve the IVP given by $\frac{dy}{dx} = 0.2y - 1, y(0) = y_0$, and explain how the value of y_0 impacts the long-term behavior of the solution (as $x \to \infty$).

Problem 3. Find the general solution of the following DE, being sure to specify the values of any arbitrary constants in your final answer.

$$\frac{dy}{dx} = xy^2 + x$$

Further practice and reference:

- Sec 11.3-11.7 of Hughes-Hallett Calculus
- Sec 7.4 of Active Calculus
- Sec 4.3 of OpenStax Calculus, vol. 2
- Math 111L Modeling Practice

6.3 Slope Fields & Equilibria

Learning objectives

6. Solve and approximate solutions to **differential equations** and utilize for modeling.
   (c) Use slope fields to approximate solutions to DEs and IVPs.
   (d) Find and classify equilibria of differential equations.

Problem 1. Consider the differential equation $\frac{dy}{dx} = y^2 - 1$.

(a) Find the equilibrium solutions, if any.

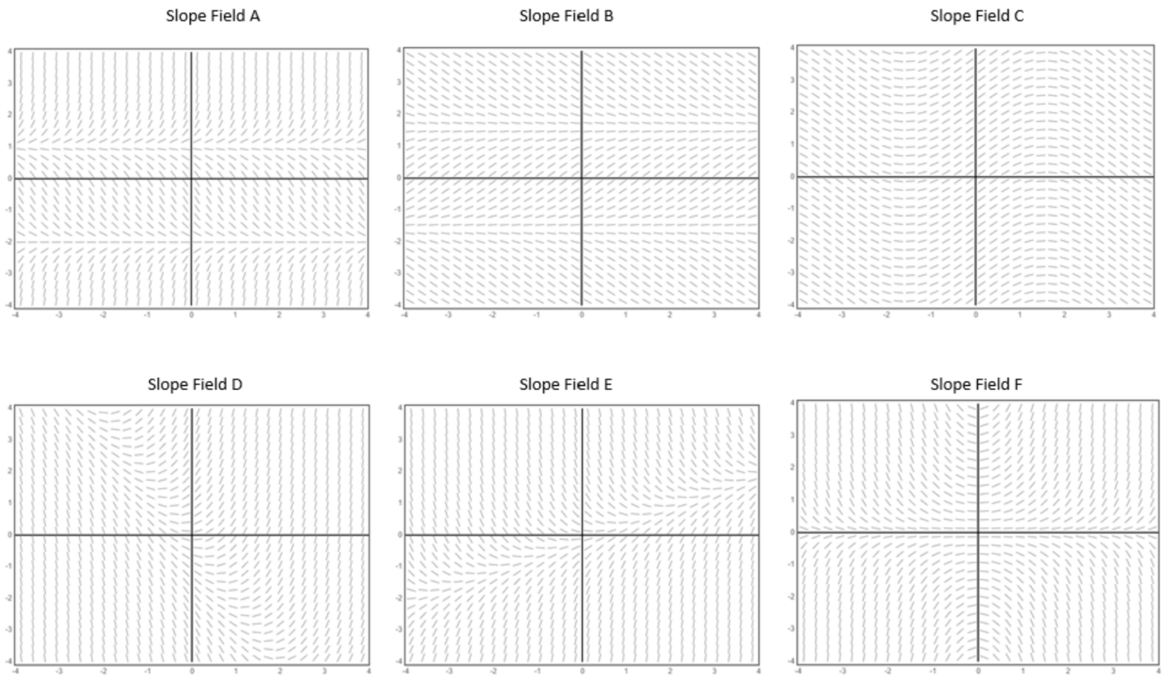
(b) Sketch an approximate slope field.

(c) Use your slope field to classify the equilibria, if there are any.

(d) Use your slope field to determine the long-term behavior (as $x \to \infty$) of solution curves passing through each of the following points:

i. $(0, 2)$
ii. $(3, 1)$
iii. $(1, 0)$
iv. $(2, -2)$

Problem 2. Consider the 6 slope fields below:



For each of the following differential equations, (1) Find the letter of the corresponding slope field, and (2) Find and classify the equilibria (that are visible in the slope field), if any.

(a) $\frac{dy}{dx} = \cos y$

(b) $\frac{dy}{dx} = x - 2y$

(c) $\frac{dy}{dx} = \cos x$

(d) $\frac{dy}{dx} = xy$

(e) $\frac{dy}{dx} = (y-1)(y+2)$

(f) $\frac{dy}{dx} = 2x + y$

Further practice and reference:

- Sec 11.1-11.2 of Hughes-Hallett Calculus
- Sec 7.2 of Active Calculus
- Sec 4.2 of OpenStax Calculus, vol. 2

6.4 Euler's Method

Learning objectives

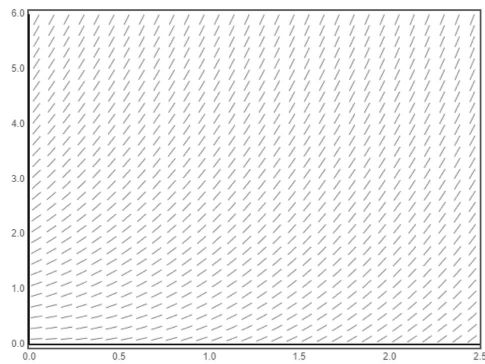
6. Solve and approximate solutions to **differential equations** and utilize for modeling.
   (c) Use slope fields and Euler's method to approximate solutions to DEs and IVPs.

Problem 1. Consider the following initial value problem:

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

(a) Use Euler's Method with a step-size of $\Delta x = 1$ to approximate $y(2)$.

(b) The slope field for the DE is shown below:



Fill-in-the-blanks: we can see from the slope field that the solution to the IVP is ________ on the interval $0 \leq x \leq 2$, so we conclude that the approximation of $y(2)$ using Euler's Method is an ________-estimate of the actual value.

Problem 2. Consider the differential equation

$$\frac{dy}{dx} = 2x(y - 1)$$

(a) Use Euler's Method with a step-size of $\Delta x = \frac{1}{2}$ to approximate $y(\frac{9}{2})$ given $y(3) = 2$.

(b) Use Euler's Method with a step-size of $\Delta x = \frac{1}{2}$ to approximate $y(\frac{9}{2})$ given $y(3) = 1$.

Further practice and reference:

- Sec 11.3 of Hughes-Hallett Calculus
- Sec 7.3 of Active Calculus
- Sec 4.2 of OpenStax Calculus, vol. 2

6.5 Modeling with DEs

Learning objectives

6. Solve and approximate solutions to **differential equations** and utilize for modeling.

   (b) Identify separable DEs and solve using the method of separation of variables.

   (c) Use slope fields and Euler's method to approximate solutions to DEs and IVPs.

   (d) Find and classify equilibria of differential equations.

   (e) Write and interpret DEs/IVPs to mathematically model real processes, including motion, exponential growth and decay, and rate in/out.

Problem 1. Let $P(t)$ represent the population of North Carolina, in millions of people, t years after 2000, and assume that the population is growing exponentially. In 2000, the population was 7.75 million people and was growing at a rate of 0.155 million people/year.

(a) What is $P'(0)$?

(b) Write an IVP satisfied by $P(t)$. (The only variables should be P and t. You can use a calculator for division if needed.)

(c) Use your model to predict the population of North Carolina currently, in 2030. Leave in exact form (or use a calculator to get a decimal approximation.)

Problem 2. A 100-gallon tank is full of saltwater. Every minute, we pump in 4 gallons of saltwater with a concentration of 1/2 pound of salt per gallon, and pump out 4 gallons of thoroughly mixed solution. Let $S(t)$ be the amount of salt in the tank, in pounds, t minutes after we start this process, and suppose there are 10 pounds of salt in the tank after 1 minute.

(a) Write an IVP solved by $S(t)$.

(b) At the instant when there are 12 pounds of salt in the tank, at what rate is the amount of salt in the tank changing? Be sure to include units.

(c) Solve your IVP to find $S(t)$.

(d) What does the model predict will be the amount of salt in the tank in the long-term, were this process to continue forever?

Problem 3. Math 111L modeling practice

Further practice and reference:

- Sec 11.5-11.7 of Hughes-Hallett Calculus
- Sec 7.5-7.6 of Active Calculus
- Sec 4.4 of OpenStax Calculus, vol. 2