Controlling Singular Values with Semidefinite Programming

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Singular Values

\[ A \]

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \sigma_3 \]

\[ \sigma_4 \]

\[ \sigma_5 \]
Singular Values

\[ A \]

\[ \sigma_{\text{min}} \quad \sigma_{\text{max}} \]
Motivation

Least Squares Conformal Mappings

As-Rigid-As-Possible

[Sorkine and Alexa 2007]

Bounded Distortion Mappings

[Lipman 2012]

[Levy et al. 2002]
Common Perspective

This Work

- Optimization of linear transformations = Matrices
- Explicitly involving singular values
  - Constraints
  - Energies
Constrained Matrix Optimization

\[
\min_{A \in \mathbb{R}^{n \times n}} f(A)
\quad s.t.
\quad g_i(A) \leq 0
\]

Energy
\[
\|A - R_A\|_F
\]

Constraints
\[
Ax = y
\quad A^T A = I
\quad \det(A) > 0
\]

\[A = (A_1, \ldots, A_m)\]
Constrained Matrix Optimization

\[ \Phi(x) = A_i x + t_i \]
Optimization of Singular Values

\[ \min_{A \in \mathbb{R}^{n \times n}} f(A) \]
\[ \text{s.t.} \]
\[ g_i(A) \leq 0 \]

- Directly optimize singular values \( \sigma_1, \ldots, \sigma_n \)
This Work

Optimization problems explicitly expressed in terms of extremal SVs

\[
\begin{align*}
\min_{A \in \mathbb{R}^{n \times n}} & \quad f(A, \sigma_{\min}(A), \sigma_{\max}(A)) \\
\text{s.t.} & \quad g_i(A, \sigma_{\min}(A), \sigma_{\max}(A)) \leq 0
\end{align*}
\]

- Challenging
  - Non-linear
  - Non-convex
  - No closed form
This Work

\[
\min_{A \in \mathbb{R}^{n \times n}} f(A, \sigma_{\min}(A), \sigma_{\max}(A)) \\
\text{s.t. } g_i(A, \sigma_{\min}(A), \sigma_{\max}(A)) \leq 0
\]

Optimization problems explicitly expressed in terms of \textbf{extremal SVs}
Key Results

\[
\min_{A \in \mathbb{R}^{n \times n}} f(A, \sigma_{\min}(A), \sigma_{\max}(A)) \\
\text{s.t. } g_i(A, \sigma_{\min}(A), \sigma_{\max}(A)) \leq 0
\]

• Can be reduced to

\[
\gamma \leq \sigma_{\min}(A) \leq \sigma_{\max}(A) \leq \Gamma
\]
Key Results

\[
\min_{A \in \mathbb{R}^{n \times n}} f(A, \sigma_{\text{min}}(A), \sigma_{\text{max}}(A))
\]
\[s.t. \quad g_i(A, \sigma_{\text{min}}(A), \sigma_{\text{max}}(A)) \leq 0\]

- Can be reduced to

\[
\gamma \leq \sigma_{\text{min}}(A) \leq \sigma_{\text{max}}(A) \leq \Gamma
\]

Semidefinite Programming

Optimal in the convex regime
Key Results

Semidefinite Programming (SDP)

- $\text{SDP} \iff \text{optimization with } \geq \text{ constraints}$
- $A \succeq 0 \iff A \text{ is positive semidefinite}$

The natural class for dealing with singular values
Bounding SVs from above

$$\sigma_{\text{max}}(A) \leq \Gamma$$

$$\begin{pmatrix} \Gamma I & A \\ A^T & \Gamma I \end{pmatrix} \succcurlyeq 0$$

convex
(operator norm)
Bounding SVs from below

\[ \gamma \leq \sigma_{\text{min}}(A) \]

non-convex

cut a convex subset?
Bounding SVs from below

\[ A \]  

Symmetric + Anti-symmetric
Bounding SVs from below

\[ \gamma \leq \sigma_{\text{min}}(A) \]

\[ \gamma \leq \sigma_{\text{min}}\left(\frac{A + A^T}{2}\right) \]

Symmetric + Anti-symmetric
Bounding SVs from below

\[ \gamma \leq \sigma_{\min}(A) \]

\[ \gamma \leq \sigma_{\min} \left( \frac{A + A^T}{2} \right) \]
Bounding SVs from below

\[ \gamma \leq \sigma_{\min}(A) \]

\[ \frac{A + A^T}{2} - \gamma I \succeq 0 \]

maximal convex subset
Maximal Convex Cover

\[ \gamma \leq \sigma_{\min}(A) \]

\[ \frac{A + A^T}{2} - \gamma I \geq 0 \]
Our Key Result

\[
\begin{pmatrix}
\Gamma I & A \\
A^T & \Gamma I
\end{pmatrix} \succeq 0
\]

\[
\frac{A + A^T}{2} - 2\gamma I \succeq 0
\]

Convex-Optimal
maximal convex cover

SDP
Semidefinite Programming

\[\gamma \leq \sigma(A) \leq \Gamma\]
Our Key Result

\[
\begin{pmatrix}
\Gamma I & A \\
A^T & \Gamma I
\end{pmatrix} \succeq 0
\]

\[
\frac{A + A^T}{2} - 2\gamma I \succeq 0
\]

[Lipman 2012]

- Specific to 2D \Rightarrow SOCP

Ours

- Any dimension \Rightarrow SDP
Optimization with SV constraints

- Start with feasible $A$
Optimization with SV constraints

• Start with feasible $A$
• Repeat:
  • Optimize over a maximal convex restriction (SDP)
Optimization with SV constraints

- Start with feasible $A$
- Repeat:
  - Optimize over a maximal convex restriction (SDP)
  - Update rotation (centering)
Optimization with SV constraints

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Every iteration energy decreases guaranteed feasibility
Optimization with SV constraints

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Optimization with SV constraints

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• Algorithm converges
• Preserves sign of determinant

$\det(A) > 0 \iff$ Orientation preserving
Applications
Shape Deformations

- As-Rigid-As-Possible [Sorkine 2007]

Unconstrained

Add constraint on SVs

\[
\frac{\sigma_{\text{max}}(A_i)}{\sigma_{\text{min}}(A_i)} \leq 2
\]

Bounded Conformal Distortion

No flips!
Extremal Quasiconformal Mappings

$$\text{minimize} \left( \max_i \frac{\sigma_{\text{max}}(A_i)}{\sigma_{\text{min}}(A_i)} \right)$$
Extremal Quasiconformal Mappings

"Most Conformal Mapping"

- Well studied in 2D [Weber et al. 2012]
- Little known in 3D…

\[
\text{minimize} \left( \max_i \frac{\sigma_{\max}(A_i)}{\sigma_{\min}(A_i)} \right)
\]

histogram of distortions
Extremal Quasiconformal Mappings

Approximate solution using our framework
Extremal Quasiconformal Mappings
Non-Rigid 3D ICP
Non-Rigid 3D ICP

Regularization

Bounded Isometric Distortion
Non-Rigid 3D ICP
Non-Rigid 3D ICP
Interpolating Rotations
Interpolating Rotations

\[
\min \left\| \mathbf{t} \right\| \quad s.t. \quad \sigma_{\min}(\mathbf{t}) \geq 1
\]

\[SO(3)\]
Averaging Rotations
Averaging Rotations
Interpolating Rotations
Concluding Remarks

• Method for optimizing

\[
\min_{A \in \mathbb{R}^{n \times n}} f(A, \sigma_{\min}(A), \sigma_{\max}(A)) \\
\text{s.t. } g_i(A, \sigma_{\min}(A), \sigma_{\max}(A)) \leq 0
\]

• Based on SDP
• Optimal in the convex regime

• Main limitation: time complexity
Thank you!

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