Cohomology of Compactifications of Locally Symmetric Spaces

IV. Micro-purity of Intersection Cohomology and Functoriality of Micro-support

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Plan of Talks

Talk 1: Introduction and Intersection Cohomology

Talk 2: Compactifications of Locally Symmetric Spaces

Talk 3: $\mathcal{L}$-modules, Micro-support, and a Vanishing Theorem

Talk 4: Micro-purity of Intersection Cohomology and Functoriality of Micro-support
Compactifications $\hat{X} \xrightarrow{\pi} X^*$ and their Links

Reductive Borel-Serre

$\hat{X} = \coprod_{P \in \mathcal{P}} X_P$

Satake

$X^* = \coprod_{R \in \mathcal{P}^*} X_{R,h}$

$(\mathcal{N}_{P\dagger} \times |\Delta_P|^{\circ})$-bundle over $X_{P\dagger,\ell}$
Recall

The Definition of an $\mathcal{L}$-module

An $\mathcal{L}$-module $\mathcal{M}$ on $\hat{X}$ consists of

- for all $P \in \mathcal{P}$, a graded algebraic representation $E_P$ of $L_P$;

- for all $P \leq Q$, degree 1 morphisms $f_{PQ} : H(n^Q_P; E_Q) \xrightarrow{[1]} E_P$.

This data must satisfy the condition that for all $P \leq R$,

$$\sum_{P \leq Q \leq R} f_{PQ} \circ H(n^Q_P; f_{QR}) = 0.$$ 

An $\mathcal{L}$-module on a locally closed union of strata $W$ is defined similarly.
Recall the

**Micro-support SS(\mathcal{M}) of an \mathcal{L}-module \mathcal{M}**

\[ V = V|_{M_P} \otimes \xi_V \] an irreducible representation of \( L_P = M_PA_P \).

Define \( Q_V \geq P \) such that

\[ \Delta_{P}^{Q_V} = \{ \alpha \in \Delta_P \mid (\xi_V + \rho, \alpha) < 0 \} \].

**Simplifying Assumption:** \( (\xi_V + \rho, \alpha) \neq 0 \) for all \( \alpha \).

**Define** \( SS_{\text{ess}}(\mathcal{M}) := \) the set of those \( V \) (for all \( P \in \mathcal{P} \)) such that

(i) \( (V|_{M_P})^* \cong V|_{M_P} \), and  
(ii) \( H(i_P^*\hat{i}_{Q_V}^! \mathcal{M})_V \neq 0 \).

and

\[ c(V; \mathcal{M}) \leq d(V; \mathcal{M}) \]

to be the least and greatest degrees where \( H(i_P^*\hat{i}_{Q_V}^! \mathcal{M})_V \neq 0 \).
In this final talk we will

(i) define the intersection cohomology $\mathcal{L}$-module $\mathcal{I}_p C(\widehat{X}; E)$;

(ii) state and discuss the **micro-purity** theorem for $\mathcal{I}_p C(\widehat{X}; E)$ which asserts that $SS_{ess}(\mathcal{I}_p C(\widehat{X}; E)) = \{E\}$;

(iii) briefly state a **functoriality** theorem for micro-support, which bounds $SS_{ess}(k^* \mathcal{M})$ and $SS_{ess}(k^! \mathcal{M})$ in terms of $SS_{ess}(\mathcal{M})$.

Together with the **vanishing** theorem from last time, this will furnish a proof of the generalized Goresky-MacPherson-Rapoport conjecture.
Truncation Functors

Recall Deligne’s formula for the intersection cohomology sheaf:

\[ \mathcal{I}_p \mathcal{C}(Z; \mathbb{E}) := \tau^{\leq p(d)} Rj_d^* \cdots \tau^{\leq p(3)} Rj_3^* \tau^{\leq p(2)} Rj_2^* \mathbb{E} . \]

In order to define an intersection cohomology \( \mathcal{L} \)-module, we need to define \( \tau^{\leq n} \mathcal{M} \) for an \( \mathcal{L} \)-module \( \mathcal{M} \).

**Example:** \( \mathcal{M} = i_{G*} E \). The local cohomology at \( P \) of \( i_{G*} E \) is \( H(\mathfrak{n}_P; E) \) and we want to define \( \tau^{\leq n} i_{G*} E \) so that the local cohomology at \( P \) becomes \( \tau^{\leq n} H(\mathfrak{n}_P; E) \).
Recall the truncation functor for complexes of sheaves:

\[
\tau^{\leq n}S := \cdots \xrightarrow{d_{n-2}} S^{n-1} \xrightarrow{d_{n-1}} \ker d_n \xrightarrow{d_n} 0 \rightarrow \cdots .
\]

It is characterized by the fact that \( \tau^{\leq n}S \to S \) induces

\[
H^j(\tau^{\leq n}S) \cong \begin{cases} 
H^j(S) & \text{for } j \leq n, \\
0 & \text{for } j > n. 
\end{cases}
\]

There is also an upper truncation functor for complexes of sheaves:

\[
\tau^{> n}S := \cdots \xrightarrow{} 0 \to S^{n+1}/\text{Im } d_n \xrightarrow{d_{n+1}} S^{n+2} \xrightarrow{d_{n+2}} \cdots .
\]

which is characterized by the fact that \( S \to \tau^{> n}S \) induces

\[
H^j(\tau^{\leq n}S) \cong \begin{cases} 
0 & \text{for } j < n, \\
H^j(S) & \text{for } j \geq n. 
\end{cases}
\]
This definition of $\tau^{\leq n} S$ cannot be used to define $\tau^{\leq n} i_{G*} E$ for $L$-modules; we cannot change $E$. However we can set $E_P = \tau^{>n} H(n_P; E)[-1]$, $f_{PP} = 0$, and let

$$f_{PG}: H(n_P; E) \rightarrow \tau^{>n} H(n_P; E)[-1]$$

be the natural degree 1 projection. Clearly the local cohomology at $P$ is now $\tau^{\leq n} H(n_P; E)$.

The idea is that we have used a shifted “mapping cone” to truncate the local cohomology: if $f: (C, d_C) \rightarrow (D, d_D)$ is a map of complexes, the mapping cone $M(f)$ is the complex defined by

$$\left( C[1] \oplus D, \begin{pmatrix} d_C [1] & 0 \\ f & d_D \end{pmatrix} \right).$$
In general the truncation $\tau^{\leq n} S$ of a complex of sheaves is quasi-isomorphic to the shifted mapping cone

$$M(S \to \tau^{>n} S)[-1].$$

This shows how to define $\tau^{\leq n} M$ for a general $\mathcal{L}$-module $M$. To truncate the local cohomology at $P$, define

$$\tau_{P}^{\leq n} M := M(\mathcal{M} \to i_{P*} \tau^{>n} i_{P*} \mathcal{M})[-1].$$

Then $\tau^{\leq n}$ is defined as the composition over all $P \in \mathcal{P}$,

$$\tau_{P_{N}}^{\leq n} \circ \tau_{P_{N-1}}^{\leq n} \circ \cdots \circ \tau_{P_{1}}^{\leq n} \circ \tau_{P_{0}}^{\leq n},$$

where if $i < j$, then $r(P_{i}) \leq r(P_{j})$.

It is easy to check that there is a natural quasi-isomorphism

$$S(\tau^{\leq n} M) \cong \tau^{\leq n} S(M).$$
The Intersection Cohomology $\mathcal{L}$-module

Define

$$j_P : W \setminus X_P \hookrightarrow W$$

where $W$ is any union of strata containing $X_P$. Also define

$$p(Q) := p(\text{codim}_\hat{X} X_Q) = p(\dim n_Q + \#\Delta_Q) \quad \text{for } Q \in \mathcal{P}.$$

Then the intersection cohomology $\mathcal{L}$-module is

$$\mathcal{I}_p C(\hat{X}; E) := \tau^{\leq p(P_N)} j_{P_N*} \cdots \tau^{\leq p(P_2)} j_{P_2*} \tau^{\leq p(P_1)} j_{P_1*} E,$$

We illustrate the procedure on the following slides in the case $\mathcal{P} = \{P < Q_1, Q_2 < G\}$. 
E

E
$j_{Q_1}^* E$

$H(n_{Q_1}; E)$

$E$
\[ \tau \leq \rho(Q_1) j_{Q_1}^* E \]

\[ \tau > \rho(Q_1) H(n_{Q_1}; E)[-1] \]

\[ H(n_{Q_1}; E) \]
\( \tau \leq p(Q_2) j_{Q_2} \) \( \tau \leq p(Q_1) j_{Q_1} E \)

\( \tau > p(Q_1) H(n_{Q_1}; E)[-1] \)

\( \tau > p(Q_2) H(n_{Q_2}; E)[-1] \)

\( H(n_{Q_1}; E) \)

\( H(n_{Q_2}; E) \)

\( E \)
\[ j_P*\tau \leq p(Q_2) j_{Q_2}*\tau \leq p(Q_1) j_{Q_1}*E \]

\[ H(n_{Q_1}^{Q_1}; \tau \geq p(Q_1) H(n_{Q_1}; E))[-1] \]

\[ \tau \geq p(Q_1) H(n_{Q_1}; E)[-1] \]

\[ H(n_{Q_1}; E) \]

\[ H(n_{P}; E) \]

\[ H(n_P^{Q_2}; \tau \geq p(Q_2) H(n_{Q_2}; E))[-1] \]

\[ \tau \geq p(Q_2) H(n_{Q_2}; E)[-1] \]

\[ H(n_{Q_2}; E) \]

\[ E \]
For the final step, note that the local cohomology complex at $P$ of the preceding $\mathcal{L}$-module is

$$(C, d_C) := \begin{array}{c}
H(n_{P}^{Q_{1}}; \tau^{>p(Q_{1})}H(n_{Q_{1}}; E))[−1] \\
H(n_{P}^{Q_{2}}; \tau^{>p(Q_{2})}H(n_{Q_{2}}; E))[−1] \\
H(n_{P}; E)
\end{array}$$

This is the complex which computes the intersection cohomology of the link at $P$.

Thus the final $\mathcal{L}$-module $\mathcal{I}_P C(\hat{X}; E)$ is . . .
\[ \mathcal{I}_p \mathcal{C}(\hat{X}; E) \]

\[ H(n_{Q_1}^P; \tau > p(Q_1) H(n_{Q_1}; E))[-1] \]

\[ \tau > p(P)(C, d_C)[-1] \]

\[ H(n_{P}^Q; \tau > p(Q_2) H(n_{Q_2}; E))[-1] \]

\[ \tau > p(Q_1) H(n_{Q_1}; E)[-1] \]

\[ H(n_{P}; E) \]

\[ H(n_{Q_1}; E) \]

\[ H(n_{Q_2}; E) \]

\[ \tau > p(Q_2) H(n_{Q_2}; E)[-1] \]
Micro-support of Intersection Cohomology

Micro-support is not always so easy to compute. The following is a deep combinatorial result.

**Micro-Purity Theorem (S., 2001).** Let $p$ be a middle perversity. Assume the $\mathbb{Q}$-root system of $G$ does not contain a factor of type $D_n$, $E_n$, or $F_4$. Let $p$ be a middle perversity. If $(E|_{M_G})^* \cong \overline{E|_{M_G}}$, then $SS_{\text{ess}}(\mathcal{I}_p\mathcal{C}(\widehat{X}; E)) = \{E\}$.

- We actually can prove a more general theorem regarding the entire micro-support $SS(\mathcal{I}_p\mathcal{C}(\widehat{X}; E))$ and without assuming $(E|_{M_G})^* \cong \overline{E|_{M_G}}$.

- The condition on $\mathbb{Q}$-type should be able to be removed.

- The analogous theorem for weighted cohomology is true (without assumption on the $\mathbb{Q}$-root system) and not that difficult.
Let $E$ be irreducible with highest weight $\lambda$. The only possible candidates for elements of $SS_{\text{ess}}(\mathcal{I}_p C(\widehat{X}; E)) = \{E\}$ are modules

$$H^\ell(w)(n_P; E)_w = V_{w(\lambda + \rho) - \rho} \quad \text{for all } w \in W_P, \ P \in \mathcal{P}$$

such that $\tau_P(w(\lambda + \rho)|_{\gamma_{MP}}) = w(\lambda + \rho)|_{\gamma_{MP}}$.

The assertion of the theorem amounts to a subtle relationship

minimal lengths in the cosets $W^R w$ for all $R \geq P \quad \longleftrightarrow \quad \text{geometry of } w(\lambda + \rho) \text{ vis à vis the roots } \alpha \in \Delta_P$

Such relationships are important in Goersky and MacPherson's (2003) topological trace formula and in the representation of cohomology classes by Eisenstein series (Schwermer, 1994), it is likely that the micro-purity theorem will have applications beyond the Goersky-MacPherson-Rapoport conjecture.
What is the theorem saying?

Since $i^*_Gi^!_G\mathcal{I}_pC(\widehat{X}; E) = E$, we certainly have $E \in \text{SS}_{\text{ess}}(\mathcal{I}_pC(\widehat{X}; E))$ if $(E|_0)_G^* \cong \overline{E|_0}_G$.

The theorem is asserting that this is all: for every proper $P \in \mathcal{P}$ with parabolic rank $r(P) = 1, 2, \ldots$, there is no irreducible $L_P$-module $V \in \text{SS}_{\text{ess}}(\mathcal{I}_pC(\widehat{X}; E))$.

To see what is happening, we will discuss proofs for $r(P) = 1$ and 2 in detail. These arguments do not generalize however; we will give a brief indication regarding the general proof.
Case $r(P) = 1$

Consider $V = V_{w(\lambda + \rho) - \rho}$. By definition of $Q_V$,

$$(w(\lambda + \rho), \alpha) < 0 \implies Q_V = G,$$

$$(w(\lambda + \rho), \alpha) > 0 \implies Q_V = P.$$ 

On the other hand, since $E_P = \tau^\geq n H(n_P; E)[-1]$, we have

$$H(i_P^* i_Q^! \mathcal{I}_P C(\widetilde{X}; E)) = \begin{cases} 
\tau^\leq p(P) H(n_P; E) & \text{if } Q = G, \\
\tau^\geq p(P) H(n_P; E)[-1] & \text{if } Q = P.
\end{cases}$$

This is zero

when $\ell(w) > p(P)$ and $Q_V = G$, or
when $\ell(w) \leq p(P)$ and $Q_V = P$. 
Recall the middle perversities are
\[ n(k) = \left\lfloor \frac{(k - 1)}{2} \right\rfloor \quad \text{and} \quad m(k) = \left\lfloor \frac{(k - 2)}{2} \right\rfloor \]

and hence
\[ p(P) = p(\dim n_P + \# \Delta_P) = \begin{cases} \left\lfloor (\dim n_P - 1)/2 \right\rfloor & p = m, \\ \left\lfloor (\dim n_P)/2 \right\rfloor & p = n. \end{cases} \]

Thus the micro-purity theorem is essentially the following version of the **Basic Lemma**:

If \((V_w(\lambda + \rho)|_{M_P})^* \cong V_w(\lambda + \rho)|_{M_P}\) then
\[ (w(\lambda + \rho), \alpha) \leq 0 \quad \implies \quad \ell(w) \geq \frac{1}{2} \dim n_P, \]
\[ (w(\lambda + \rho), \alpha) \geq 0 \quad \implies \quad \ell(w) \leq \frac{1}{2} \dim n_P. \]

Furthermore if \((E|_{M_G})^* \cong E|_{M_G}\), then
\[ \ell(w) = \frac{1}{2} \dim n_P \quad \implies \quad (w(\lambda + \rho), \alpha) = 0. \]
Let us sketch the proof of the lemma.

Pick a fundamental $\theta$-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}_P$ and a positive system of roots $\Phi^+$, consisting of the roots in $\mathfrak{n}_{P\mathbb{C}}$ together with a $\theta$-stable set of roots in $\mathfrak{l}_{P\mathbb{C}}$. With these choices, $\tau_P = \theta$. Decompose

$$\mathfrak{h} = \mathfrak{h}_{M_P,1} + (\mathfrak{h}_{M_P,-1} + \mathfrak{a}_P)$$

according to the $\pm 1$-eigenspaces of $\theta$.

It is standard that

$$\ell(w) := \#\{\gamma \in \Phi^+ | w^{-1}\gamma < 0\} = \#\{\gamma \in \Phi^+ | (w(\lambda + \rho), \gamma) \leq 0\}.$$

and for $w \in W_P$, such roots are in $\mathfrak{n}_{P\mathbb{C}}$. Note that the roots of $\mathfrak{n}_{P\mathbb{C}}$ are preserved under the involution $\gamma \mapsto \bar{\gamma}$. 
Since $\tau_P = \theta$, the hypothesis $(V|_{M_P})^* \cong \overline{V|_{M_P}}$ implies that

$$w(\lambda + \rho)|_{\eta_{M_P, -1}} = 0.$$ 

Thus

$$(w(\lambda + \rho), \gamma) = (w(\lambda + \rho), \gamma|_{\eta_{M_P, 1}}) + (w(\lambda + \rho), \gamma|_{a_P}).$$

If $(w(\lambda + \rho), \alpha) \leq 0$, the second term is nonpositive. On the other hand, if we replace $\gamma$ by $\bar{\gamma}$, the first term is negated. Thus the equation is negative for all real roots and for at least one of every pair $\{\gamma, \bar{\gamma}\}$ of complex roots. This proves the first part of the lemma; the second part is similar.
For the third part, assume $\ell(w) = \frac{1}{2} \dim \mathfrak{p}_P$. By the previous calculation, for each root $\gamma$ in $\mathfrak{p}_P \mathbb{C}$, exactly one of $\{w^{-1}\gamma, w^{-1}\bar{\gamma}\}$ is negative. It follows that the positive system $w\Phi^+$ is $\theta$-stable and thus $\tau_G = \theta$.

Together with the hypothesis $(E|_{M_G})^* \cong \overline{E|_{M_G}}$, this implies

$$w(\lambda + \rho)|_{h_{M_P,-1} + a_P} = 0.$$
Bidegree in $H(n_P; E)$

Now consider $r(P) \geq 2$. Recall that for $P \leq Q$

$$H(n_P; E) \cong H(n^Q_P; H(n_Q; E))$$

is bigraded. Write the bidegree as $(\ell^Q, \ell_Q)$ and define truncation by bidegree functors $\tau^{\ell_Q \leq n}$ and $\tau^{\ell_Q > n}$.

One may check that


and thus $w \in W_P$ may be decomposed as $w^Q w_Q$. It is easy then to verify that

$$H(n_P; E)_w \cong H(n^Q_P; H(n_Q; E)_{w_Q})_{w_Q}$$

and so the bidegree of $H(n_P; E)_w$ is

$$(\ell^Q(w), \ell_Q(w)) := (\ell(w^Q), \ell(w_Q)).$$
To calculate bidegree, note that

\[
\ell(w) = \#\{\gamma \in \Phi^+ \mid (w(\lambda + \rho), \gamma) \leq 0\},
\]

\[
\ell_Q(w) = \# \text{ of such roots occurring in } n_{QC},
\]

\[
\ell_Q^Q(w) = \# \text{ of such roots occurring in } \tilde{n}_{PC}^Q.
\]

**Examples:**

(a) \(\Delta_P = \{\alpha_1, \alpha_2\}\),
\(\Delta_P^Q = \{\alpha_1\}\)

(b) \(\Delta_P = \{\alpha_1, \ldots, \alpha_5\}\),
\(\Delta_P^Q = \{\alpha_2, \alpha_4, \alpha_5\}\)
Case $r(P) = 2$

Recall the complex computing intersection cohomology of the link at $P$ is

$$H(n_{Q_1}^P; \tau^{>p(Q_1)} H(n_{Q_1}; E))[-1]$$

and so the link intersection cohomology is

$$\tau^{\ell Q_1 \leq p(Q_1)} \tau^{\ell Q_2 \leq p(Q_2)} H(n_P; E) \bigoplus \tau^{\ell Q_1 > p(Q_1)} \tau^{\ell Q_2 > p(Q_2)} H(n_P; E)[-1].$$

It helps (especially in higher parabolic rank) to view the calculation more geometrically . . .
Write the $H(n_P; E)_w$-isotypical component of the link intersection cohomology at $P$ as

$$H(n_P; E)_w \otimes I_{p_w}H(|\Delta_P|; \mathbb{Z}),$$

where $p_w$ is the \textit{w-shifted generalized perversity}

$$p_w(Q) := p(Q) - \ell_Q(w) = p(\dim n_Q + \#\Delta_Q) - \ell_Q(w).$$

\textbf{Example:} For $r(P) = 2$, set $B_w = \{ \alpha_i \in \Delta_P \mid \ell_{Q_i}(w) > p(Q_i) \}$. Then:

$$I_{p_w}C(|\Delta_P|; \mathbb{Z}):\begin{array}{c}
\emptyset \quad \{\alpha_1\} \quad \{\alpha_2\} \quad \{\alpha_1, \alpha_2\}
\end{array}
B_w:\begin{array}{c}
\emptyset \quad \{\alpha_1\} \quad \{\alpha_2\} \quad \{\alpha_1, \alpha_2\}
\end{array}
I_{p_w}H(|\Delta_P|; \mathbb{Z}):\begin{array}{c}
\mathbb{Z} \quad 0 \quad 0 \quad \mathbb{Z}[-1]
\end{array}

Note in this case that $I_{p_w}H(|\Delta_P|; \mathbb{Z}) = H(|\Delta_P|, B_w)$. 
But we don’t want the link intersection cohomology. Instead we want

\[ H(i_P^*i_Q^!\mathcal{I}_P\mathcal{C}(\widehat{X}; E)) \cong \bigoplus_w H(n_P; E)_w \otimes I_{p_w} H_{c(|\Delta_P^Q|)} \]

(cohomology with supports). We calculate

\[
H(i_P^*i_Q^!\mathcal{I}_P\mathcal{C}(\widehat{X}; E)) = \\
\begin{cases}
\tau \leq p(P) \left( \tau^{\ell Q_1 \leq p(Q_1)} \tau^{\ell Q_2 \leq p(Q_2)} H(n_P; E) \oplus \tau^{\ell Q_1 > p(Q_1)} \tau^{\ell Q_2 > p(Q_2)} H(n_P; E)[-1] \right) & \text{for } Q = G, \\
\tau > p(P) \left( \tau^{\ell Q_1 \leq p(Q_1)} \tau^{\ell Q_2 \leq p(Q_2)} H(n_P; E) \oplus \tau^{\ell Q_1 > p(Q_1)} \tau^{\ell Q_2 > p(Q_2)} H(n_P; E)[-1] \right)[-1] & \text{for } Q = P.
\end{cases}
\]

Just like the \( r(P) = 1 \) case, one can show using a version of the Basic Lemma that if \( Q_V = G \) or \( P \), then \( V \notin SS_{ess} \). (However such \( V \) may occur in \( SS \) if \( P \) contains a fundamental parabolic \( \mathbb{R} \)-subgroup.)
But **unlike** the parabolic rank 1 case, there are two more cases to consider, $Q_V = Q_1$ and $Q_V = Q_2$. We consider $Q_V = Q_1$.

Note that $i_P^* i_{Q_1}^! \mathcal{I}_p \mathcal{C}(\hat{X}; E)$ is the complex

$$\tau^{\ell_{Q_1} > p(Q_1)} H(n_P; E)[-1] \rightarrow \tau^{> p(P)} (C, d_C)[-1]$$

where $(C, d_C)$ is the link intersection cohomology complex at $P$:

$$\tau^{\ell_{Q_1} > p(Q_1)} H(n_P; E))[-1]$$

$$H(n_P; E)$$

$$\tau^{\ell_{Q_2} > p(Q_2)} H(n_P; E))[-1]$$
We compute

\[ H(i_P^* i_{Q_1}^! I_p C(\hat{X}; E)) = \tau^{\ell Q_1 > p(Q_1)} \tau^{\ell Q_2 \leq p(Q_2)} H(n_P; E)[-1] \bigoplus \]
\[ \tau^{\leq p(P)} \left( \tau^{\ell Q_1 > p(Q_1)} \tau^{\ell Q_2 > p(Q_2)} H(n_P; E)[-1] \right) \bigoplus \]
\[ \tau^{> p(P)} \tau^{\ell Q_1 \leq p(Q_1)} \tau^{\ell Q_2 \leq p(Q_2)} H(n_P; E)[-1]. \]

- The first term are those classes in \( \tau^{\ell Q_1 > p(Q_1)} H(n_P; E)[-1] \) which map to 0 in the link cohomology,

- the next term are those classes which map to nonzero elements in the link cohomology which are not truncated at \( P \), and

- the third term are those link cohomology classes being truncated at \( P \) which do not come from \( \tau^{\ell Q_1 > p(Q_1)} H(n_P; E)[-1] \).
If $Q_V = Q_1$ we can show $H(i_P^* i_{Q_1}^! \mathcal{I}_p \mathcal{C}(\tilde{X}; E))_V = 0$ using

**Proposition.** Assume $(V_{w(\lambda+\rho)-\rho}|_{M_P})^* \cong \overline{V_{w(\lambda+\rho)-\rho}|_{M_P}}$ and that

$$(w(\lambda + \rho), \alpha_1) \leq 0 \quad \text{and} \quad (w(\lambda + \rho), \alpha_2) \geq 0.$$ 

(i) If $\ell_{Q_1}(w) \geq (\dim \mathfrak{n}_{Q_1})/2$ then $\ell(w) \geq (\dim \mathfrak{n}_P)/2$.

(ii) If $\ell_{Q_2}(w) \leq (\dim \mathfrak{n}_{Q_2})/2$ then $\ell(w) \leq (\dim \mathfrak{n}_P)/2$.

If either hypothesis is a strict inequality, the corresponding conclusion is also strict.

**Idea of the proof of (i):**

Re-interpret $V = H^{\ell(w)}(\mathfrak{n}_P; E)_w$ as $H^{\ell_{Q_1}(w)}(\mathfrak{n}_{Q_1}; F)_{wQ_1}$ where $F = H^{\ell_{Q_1}(w)}(\mathfrak{n}_{Q_1}; E)_{wQ_1}$ and apply the Basic Lemma to the parabolic subgroup $P/N_{Q_1}$ of $L_{Q_1}$. \hfill \Box
**Case** \( r(P) = 3 \)

One can treat this by the same ideas but there are more cases:

\[
\begin{align*}
I_{pw} C(|\Delta_P|; \mathbb{Z}) : & \quad \triangle \quad \triangle \quad \bullet \quad \bullet \quad \bullet \\
I_{pw} H(|\Delta_P|; \mathbb{Z}) : & \quad \mathbb{Z} \quad 0 \quad \mathbb{Z}[-1] \quad \mathbb{Z}^2[-1] \\
\end{align*}
\]

\[
\begin{align*}
I_{pw} C(|\Delta_P|; \mathbb{Z}) : & \quad \triangle \quad \triangle \quad \bullet \quad \bullet \quad \bullet \\
I_{pw} H(|\Delta_P|; \mathbb{Z}) : & \quad 0 \quad \mathbb{Z}[-1] \quad \mathbb{Z}[-1] \quad 0 \\
\end{align*}
\]

\[
\begin{align*}
I_{pw} C(|\Delta_P|; \mathbb{Z}) : & \quad \triangle \quad \triangle \quad \bullet \quad \bullet \quad \bullet \\
I_{pw} H(|\Delta_P|; \mathbb{Z}) : & \quad \mathbb{Z}^2[-1] \quad \mathbb{Z}[-1] \quad 0 \quad \mathbb{Z}[-2] \\
\end{align*}
\]
**Case** $r(P)$ general

The argument using the Basic Lemma to handle the cases $Q_V = P$ or $Q_V = G$ applies in general.

But for $r(P) > 3$, the preceding methods will not work for $P < Q_V < G$ unless we knew more precise information about $I_{pw} H(|\Delta_R|)$ for $R > P$. Thus we would like an inductive argument.

The difficulty is that the hypothesis

\[ V|_{M_P} = H^{\ell(w)}(n_P; E)_w|_{M_P} \]

is conjugate self-contragredient

is **not** preserved under the induction where we replace $P$ by $R > P$ and $H^{\ell(w)}(n_P; E)_w$ by $H^{\ell(w_R)}(n_R; E)_w_R$.

The solution is to find new hypotheses that are preserved under such induction . . .
A variant of the basic lemma implies that for all $A_P$-weights $\alpha$ occurring in $n_P$, 

\[
\begin{cases}
\ell_\alpha(w) \geq (\dim n_{P,\alpha})/2 & \text{if } \alpha \text{ occurs in } n_{P}^{Q_V}, \\
\ell_\alpha(w) \leq (\dim n_{P,\alpha})/2 & \text{if } \alpha \text{ occurs in } n_{P}^{S_V}, 
\end{cases}
\]

where $S_V$ is the parabolic $\mathbb{Q}$-subgroup complementary to $Q_V$, that is, $\Delta_P^{S_V} := \Delta_P \setminus \Delta_P^{Q_V}$. These conditions are still not preserved by induction from $P$ to $R$.

However we can prove that there exists $T \geq P$ (depending on $w$) with $r(T) = 1$ or 2 such that 

\[
\begin{cases}
\ell_\alpha(w) \geq (\dim n_{\alpha})/2 & \text{if } \alpha \text{ occurs in } n_{T}^{Q_V \setminus T}, \\
\ell_\alpha(w) \leq (\dim n_{\alpha})/2 & \text{if } \alpha \text{ occurs in } n_{T}^{S_V \setminus T}.
\end{cases}
\]

These conditions are now preserved by induction from $P$ to $R$ provided $R \leq T$.

The proof uses the theory of quasi-minuscule representations.
Example of $n_{P}^{Q_{V}}$ versus $n_{T}^{Q_{V} \lor T}$ and $n_{P}^{S_{V}}$ versus $n_{T}^{S_{V} \lor T}$

$(\Delta_{P} = \{\alpha_{1}, \ldots, \alpha_{5}\}, \Delta_{P}^{Q_{V}} = \{\alpha_{2}, \alpha_{4}, \alpha_{5}\}, \text{and } \Delta_{P}^{T} = \{\alpha_{1}, \alpha_{4}, \alpha_{5}\})$
To finish the proof of micro-purity we must show

\[ I_{pw} H_c(|\Delta^Q_P|) = 0 \]

which is equivalent in high degree to

\[ I_{pw} H(|\Delta_P| \setminus |\Delta^Q_P|) = 0. \]

The argument has two parts:

(i) Vanishing for the cohomology in \(|\Delta_P| \setminus |\Delta^Q_P|\) is proved with a Mayer-Vietoris spectral sequence.
(ii) Vanishing for the cohomology supported on \(|\Delta^Q_{P/V} \setminus \Delta^Q_P|\) is proved with a Fary spectral sequence and induction from \(P\) to \(R \leq T\).

\[
U_{\alpha_1} \cap |\Delta^Q_P \cup \{\alpha_1\}|
\]

\[
U_{\alpha_1} \cap U_{\alpha_2} \cap |\Delta^Q_P \cup \{\alpha_1, \alpha_2\}|
\]

\[
U_{\alpha_2} \cap |\Delta^Q_P \cup \{\alpha_2\}|
\]

The Fary spectral sequence
Functoriality of Micro-support

**Functoriality Theorem.** Let $\mathcal{M}$ be an $\mathcal{L}$-module with $SS_{\text{ess}}(\mathcal{M}) = \{E\}$ and let $x \in X_{R,h}$ be a point of a stratum of a Satake compactification $X^*$ with real equal-rank boundary components. Let $\pi: \hat{X} \to X^*$ be Zucker’s projection and let $k: \pi^{-1}(x) \cong \hat{X}_{R,\ell} \hookrightarrow \hat{X}$ denote the inclusion. Then

\[
\begin{align*}
    d(k^*\mathcal{M}) &\leq \frac{1}{2} \text{codim } X_{R,h} - \#\Delta_R, \text{ and} \\
    c(k^!\mathcal{M}) &\geq \frac{1}{2} \text{codim } X_{R,h} + \#\Delta_R.
\end{align*}
\]

This theorem is actually a special case of a more general result on the functoriality of micro-support: for $\mathcal{M}$ an arbitrary $\mathcal{L}$-module and $X^*$ a Satake compactification with real equal-rank boundary components, the theorem gives a bound on $SS(k^*\mathcal{M})$ and $SS(k^!\mathcal{M})$ in terms of $SS(\mathcal{M})$. 
Proof of the Goresky-MacPherson-Rapoport Conjecture

Let $\mathcal{M}$ be an $\mathcal{L}$-module for which $\text{SS}_{\text{ess}}(\mathcal{M}) = \{E\}$. For example, this holds for intersection cohomology by the Micro-Purity Theorem. Assume $\pi : \widehat{X} \to X^*$ is the projection onto a Satake compactification with equal-rank real boundary components. For the Main Theorem, $\pi_* \mathcal{M} = \mathcal{IC}(X^*; \mathbb{E})$, we need to check the local vanishing condition on $\pi_* \mathcal{M}$:

$$H^i(i_x^* \pi_* \mathcal{M}) = 0 \quad \text{for } x \in X_{R,h}, \ i \geq \frac{1}{2} \text{codim } X_{R,h},$$

$$H^i(i_x^! \pi_* \mathcal{M}) = 0 \quad \text{for } x \in X_{R,h}, \ i \leq \frac{1}{2} \text{codim } X_{R,h},$$

where $i_x : \{x\} \hookrightarrow X^*$ denotes the inclusion.

However $\pi^{-1}(x) \cong \widehat{X}_{R,\ell} \times \{x\}$. Since $H^i(i_x^* \pi_* \mathcal{M}) = H^i(\widehat{X}_{R,\ell}; k^* \mathcal{M})$ we can use the Vanishing Theorem to see this vanishes for $i > d(k^* \mathcal{M})$. The Functoriality Theorem now completes the proof.
The weighted cohomology sheaf \( \mathcal{WC}(\tilde{X}; E) \) also satisfies \( SS_{ess}(\mathcal{M}) = \{ E \} \); the proof is easier since we have an explicit formula for the local weighted cohomology. Then the same argument yields a new proof (and a generalization to the equal-rank case) of the main result of Goresky, Harder, and MacPherson (1994) on weighted cohomology.