Cohomology of Compactifications of
Locally Symmetric Spaces
II. Compactifications of Locally Symmetric Spaces

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Plan of Talks

Talk 1: Introduction and Intersection Cohomology

Talk 2: Compactifications of Locally Symmetric Spaces

Talk 3: $\mathcal{L}$-modules, Micro-support, and a Vanishing Theorem

Talk 4: Micro-purity of Intersection Cohomology and Functoriality of Micro-support
Recall we are interested in **locally symmetric spaces**:

\[ X = \Gamma \backslash G/KA_G = \Gamma \backslash D \]

where

- \( G \) := the real points of a connected reductive group over \( \mathbb{Q} \),
- \( K \) := a maximal compact subgroup of \( G \),
- \( A_G \) := the identity component \((\mathbb{R}^+)^s\) of a maximal \( \mathbb{Q} \)-split torus in the center of \( G \),
- \( D := G/KA_G \), the corresponding symmetric space,
- \( \Gamma := \) an arithmetic subgroup of \( G \).

**Example:** \( X = \text{SL}(2,\mathbb{Z}) \backslash \text{GL}(2,\mathbb{R})/\text{O}(2)\mathbb{R}^+ \), the moduli space of complex elliptic curves.

In general \( X \) is noncompact and we have various ways to compactify.
Compactifications of Locally Symmetric Spaces

X is "topological": add boundary (or corners); does not change homotopy type.

X is "geometric": collapse boundary strata so that metric extends to a non-degenerate metric on the strata.

X is due to Zucker (1983).

X (in the Hermitian Baily-Borel-Satake case) is "complex-analytic" and "algebraic": collapse strata further so that complex structure extends.

X is a projective algebraic variety defined over a number field (Baily and Borel, 1966). The description as a quotient of X is due to Zucker (1983).
Compactifications of Locally Symmetric Spaces

$\overline{X}$

Borel-Serre

- $\overline{X}$ is “topological”: add boundary (or corners); does not change homotopy type.
Compactifications of Locally Symmetric Spaces

\[ \overline{X} \quad \text{Borel-Serre} \quad \rightarrow \quad \widehat{X} \quad \text{Reductive Borel-Serre} \]

- \( \overline{X} \) is “topological”: add boundary (or corners); does not change homotopy type.
- \( \widehat{X} \) is “geometric”: collapse boundary strata so that metric extends to a non-degenerate metric on the strata. \( \widehat{X} \) is due to Zucker (1983).
Compactifications of Locally Symmetric Spaces

\[
\begin{array}{ccc}
\overline{X} & \xrightarrow{\text{Borel-Serre}} & \widehat{X} & \xrightarrow{\pi} & X^* \\
\end{array}
\]

- \(\overline{X}\) is “topological”: add boundary (or corners); does not change homotopy type.
- \(\widehat{X}\) is “geometric”: collapse boundary strata so that metric extends to a non-degenerate metric on the strata. \(\widehat{X}\) is due to Zucker (1983).
- \(X^*\) (in Hermitian Baily-Borel-Satake case) is “complex-analytic” and “algebraic”: collapse strata further so that complex structure extends. \(X^*\) is a projective algebraic variety defined over a number field (Baily and Borel, 1966). The description as a quotient of \(\widehat{X}\) is due to Zucker (1983).
Example: \( \text{SL}(2, \mathbb{Z}) \backslash H \)

Near “infinity”, \( \text{SL}(2, \mathbb{Z}) \) acts via \( \{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \} \):

\[
\begin{array}{c}
\text{x-axis} \times \{ \infty \} \\
\text{x-axis} \times \{ b \}
\end{array}
\]

Here \( \mathbb{S}^1 \) is the \( x \)-axis modulo \( \mathbb{Z} \).
Example: $\text{SL}(2, \mathbb{Z}) \backslash H$

Near "infinity", $\text{SL}(2, \mathbb{Z})$ acts via $\left\{ \left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right) \mid n \in \mathbb{Z} \right\}$:

Thus

<table>
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<tr>
<td>$\overline{X}$</td>
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Here $S^1$ is the $x$-axis modulo $\mathbb{Z}$.

Metric near cusp is

$$dr^2 + e^{-2r} ds_{S^1}^2.$$
Example: Hilbert Modular Surface \( SL(2, \mathcal{O}_k) \backslash (H \times H) \)

Here \( k = \mathbb{Q}(\sqrt{d}), \quad d > 0 \). Near “infinity”, \( \text{SL}(2, \mathcal{O}_k) \) acts via

\[
\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\} a \in \mathcal{O}_k \quad \times \quad \left\{ \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \right\} u \in \mathcal{O}^\times_k
\]

\[
\mathcal{O}_k = \mathbb{Z} + \mathbb{Z} \delta
\]

\[
\mathcal{O}^\times_k = \{ u^k \mid k \in \mathbb{Z} \}
\]
Thus

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The hyperbola $y_1y_2 = b$ in the $y_1y_2$-plane becomes the $S^1$ above under the action of $\{ \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} | u \in O_k^\times \}$. The $T^2$-fibers correspond to the $x_1x_2$-plane modulo a lattice.

The metric near the cusp point is

$$dr^2 + ds_{S^1}^2 + e^{-2r} ds_{T^2}^2.$$
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The hyperbola $y_1y_2 = b$ in the $y_1y_2$-plane becomes the $S^1$ above under the action of $\left\{ \left( \begin{array}{cc} u & 0 \\ 0 & u^{-1} \end{array} \right) \mid u \in \mathcal{O}_k^\times \right\}$. The $T^2$-fibers correspond to the $x_1x_2$-plane modulo a lattice.

The metric near the cusp point is

$$dr^2 + ds_{S^1}^2 + e^{-2r}ds_{T^2}^2.$$ 

In general there are many strata; for $\overline{X}$ and $\widehat{X}$ they are indexed by parabolic $\mathbb{Q}$-subgroups $P$ mod $\Gamma$-conjugacy.

The flat $T^2$-bundle over $S^1$ will be replaced by a flat $N'_P$-bundle over $X_P$, where $X_P$ is a locally symmetric space $X_P$, and $N'_P$ is a compact nilmanifold.
Overview of Strata and Links in $\overline{X} \to \hat{X}$

$\mathcal{P} := \Gamma$-conjugacy classes of parabolic $Q$-subgroups $P$ of $G$

$r(P) :=$ parabolic rank of $P$

$|\Delta_P|^\circ :=$ open simplex of dimension $r(P) - 1$

<table>
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<th>Link$^\circ$</th>
<th>Picture</th>
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<tr>
<td>$\overline{X}$ $Y_P = \mathcal{N}'_P$-bundle over $X_P$</td>
<td>$</td>
<td>\Delta_P</td>
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<td>$\mathcal{N}'_P \times</td>
<td>\Delta_P</td>
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Overview of Strata and Links in $\overline{X} \to \widehat{X}$

$$\mathcal{P} := \Gamma\text{-conjugacy classes of parabolic } \mathcal{Q}\text{-subgroups } P \text{ of } G$$

$$r(P) := \text{parabolic rank of } P$$

$$|\Delta_P|^{\circ} := \text{open simplex of dimension } r(P) - 1$$

| Boundary stratum associated to $P \in \mathcal{P}$ | $|\Delta_P|^{\circ}$ | Picture |
|--------------------------------------------------|-----------------|---------|
| $\overline{X}$ | $Y_P = \mathcal{N}'_P\text{-bundle over } X_P$ | |
| $\widehat{X}$ | $X_P$ | $\mathcal{N}'_P \times |\Delta_P|^{\circ}$ | |

The Satake compactifications $X^*$ are more complicated . . .
Overview of Satake Compactifications $X^*$ (Satake, 1960)

- Start with “geometrically rational” representation $V$ of $G$.
- This induces $X_P = X_{P,\ell} \times X_{P,h}$ for all $P$.
- Strata of $X^*$ are the various $X_{P,h}$.
- $\pi: \tilde{X} \to X^*$ induced by projection, stratum by stratum.

Several $P \in \mathcal{P}$ can yield same $X_{P,h}$ — let $P^\dagger = $ maximal one. Strata are indexed by $\mathcal{P}^* := \{ P^\dagger \mid P \in \mathcal{P} \}$.

\[
\begin{array}{c}
X : \\
\begin{array}{c}
\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \alpha_8 \alpha_9 \\
\delta = \text{h.w. } V
\end{array}
\end{array}
\]

\[
\begin{array}{c}
X_P : \\
\begin{array}{c}
\alpha_1 \alpha_2 \alpha_4 \\
\overline{X_{P,\ell}}
\end{array} \quad \begin{array}{c}
\alpha_6 \alpha_7 \alpha_8 \alpha_9 \\
\overline{X_{P,h}}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
X_{P^\dagger} : \\
\begin{array}{c}
\alpha_1 \alpha_2 \alpha_3 \alpha_4 \\
\overline{X_{P^\dagger,\ell}}
\end{array} \quad \begin{array}{c}
\alpha_6 \alpha_7 \alpha_8 \alpha_9 \\
\overline{X_{P^\dagger,h}}
\end{array}
\end{array}
\]
Overview of Compactifications of $X$

Borel-Serre

\[ \overline{X} = \bigsqcup_{P \in \mathcal{P}} Y_P, \]

\[ \Downarrow \] \hspace{2cm} \text{Collapse } \mathcal{N}_P' \text{ fibers} \]

Reductive Borel-Serre

\[ \hat{X} = \bigsqcup_{P \in \mathcal{P}} X_P, \]

\[ \pi \Downarrow \] \hspace{2cm} \text{Project } X_P = X_{P,\ell} \times X_{P,h} \rightarrow X_{P,h} = X_{P^t,h} \]

Satake

\[ X^* = \bigsqcup_{R \in \mathcal{P}^*} X_{R,h} \]
Overview of Compactifications of $X$

Borel-Serre

\[ \overline{X} = \bigsqcup_{P \in \mathcal{P}} Y_P, \]
\[ \downarrow \quad \text{Collapse } N'_P \text{ fibers} \]

Reductive Borel-Serre

\[ \widehat{X} = \bigsqcup_{P \in \mathcal{P}} X_P, \]
\[ \pi \downarrow \quad \text{Project } X_P = X_{P,\ell} \times X_{P,h} \to X_{P,h} = X_{P^i,h} \]

Satake

\[ X^* = \bigsqcup_{R \in \mathcal{P}^*} X_{R,h} \]

When passing from $\overline{X}$ to $\widehat{X}$ to $X^*$

- strata become simpler;
- links become more complicated; and hence
- local intersection cohomology becomes more complicated
The Borel-Serre Compactification $\overline{X} = \Gamma \backslash \overline{D}$

The construction of $\overline{D}$ has 3 steps but first we need the structure of parabolic subgroups.

$P :=$ the real points of a parabolic $\mathbb{Q}$-subgroup of $G$,
$N_P :=$ the unipotent radical of $P$,
$L_P :=$ the Levi quotient $P/N_P$.

One can lift (not uniquely) $L_P$ to $\tilde{L}_P \subseteq P$ so that $P = \tilde{L}_P N_P$.

**Example:** $D =$ positive definite $n \times n$ matrices modulo homothety
Here $G = \text{GL}_n(\mathbb{R})$, $K = \text{O}(n)$, $A_G =$ positive scalar matrices.

Any $P$ is conjugate to a group of block upper-triangular matrices:

$$
\begin{pmatrix}
\text{GL}_{n_0}(\mathbb{R}) & \ast \\
\text{GL}_{n_1}(\mathbb{R}) & \ddots \\
0 & \cdots & \text{GL}_{n_r}(\mathbb{R})
\end{pmatrix}
= 
\begin{pmatrix}
\text{GL}_{n_0}(\mathbb{R}) & 0 \\
\text{GL}_{n_1}(\mathbb{R}) & \ddots \\
0 & \cdots & \text{GL}_{n_r}(\mathbb{R})
\end{pmatrix}
\begin{pmatrix}
I_{n_0} & \ast \\
I_{n_1} & \ddots \\
0 & \cdots & I_{n_r}
\end{pmatrix}
$$
$A_P := \text{the identity component } (\mathbb{R}^+)^{r(P)+s} \text{ of a maximal }$

$\mathbb{Q}\text{-split torus in center of } L_P,$

$M_P := {}^0L_P := \bigcap_{\alpha \in X(L_P)} \alpha^2.$

We have $L_P = M_PA_P$ and this lifts to $\tilde{L}_P = \tilde{M}_P\tilde{A}_P.$

Also $A_P = A_G \times \tilde{A}_P^G$ where $\tilde{A}_P^G := A_P \cap {}^0G.$

**Example:**

$$
\begin{pmatrix}
\text{GL}_{n_0}(\mathbb{R}) & 0 \\
\text{GL}_{n_1}(\mathbb{R}) & \ddots \\
0 & \text{GL}_{n_r}(\mathbb{R})
\end{pmatrix} = 
\begin{pmatrix}
\text{SL}_{n_0}^\pm(\mathbb{R}) & 0 \\
\text{SL}_{n_1}^\pm(\mathbb{R}) & \ddots \\
0 & \text{SL}_{n_r}^\pm(\mathbb{R})
\end{pmatrix} 
\begin{pmatrix}
\mathbb{R}^+I_{n_0} & 0 \\
\mathbb{R}^+I_{n_1} & \ddots \\
0 & \mathbb{R}^+I_{n_r}
\end{pmatrix}
$$

where $\text{SL}_{n_i}^\pm(\mathbb{R})$ denotes matrices with determinant $\pm1.$

The second factor is in $\tilde{A}_P^G$ if and only if it has determinant $1.$
$L_P$ acts on the Lie algebra $\mathfrak{n}_P$ of $N_P$ via lift of adjoint action.

$\Delta_P = \{\alpha_1, \ldots, \alpha_r\} := \text{simple roots}$ (indecomposable $A^G_P$-weights),
$\hat{\Delta}_P = \{\beta_1, \ldots, \beta_r\} := \text{“dual” basis.}$

For $P$ a minimal parabolic $Q$-subgroup, these are the simple roots and fundamental weights of the $Q$-root system.

Example: $a^{\alpha_i} = \lambda_i/\lambda_{i-1}$ and $a^{\beta_i} = 1/(\lambda_0 \cdots \lambda_{i-1})$, where

$$a = \begin{pmatrix} \lambda_0 \cdot I_{n_0} & 0 & & \\ \lambda_1 \cdot I_{n_1} & \ddots & \\ & \ddots & \ddots & \\ 0 & \cdots & \lambda_r \cdot I_{n_r} & \end{pmatrix}. $$
Define the strictly dominant cone

\[ A_P^{G^+} \equiv \{ a \in A_P^G \mid a^\alpha > 1 \text{ for all } \alpha \in \Delta_P \}, \]

and the negative codominant cone,

\[ -A_P^G \equiv \{ a \in A_P^G \mid a^\beta \leq 1 \text{ for all } \beta \in \tilde{\Delta}_P \}. \]

**Example:** \( P = \) minimal parabolic \( \mathbb{Q} \)-subgroup of \( GL_3(\mathbb{R}) \)
Parabolic $Q$-subgroups containing a given one

For any parabolic $Q$-subgroup $Q \supseteq P$, decompose

$$N_P = (N_P \cap \tilde{L}_Q)N_Q = \tilde{N}_P^Q N_Q.$$ 

parabolics $Q \supseteq P \leftrightarrow$ subsets $\Delta_P^Q \subseteq \Delta_P$

$$Q \leftrightarrow \{ \alpha \in \Delta_P \mid n_{P, \alpha} \subseteq \tilde{n}_P^Q \}$$

Example: $\Delta_P^Q = \{ \alpha_1 \}$

$$Q = \begin{pmatrix} & \cr & N_Q \cr \end{pmatrix}, \quad P = \begin{pmatrix} & \cr & N_P \cr \end{pmatrix}, \quad N_P = \begin{pmatrix} & \cr \tilde{N}_P^Q & \cr \end{pmatrix}$$
The Decomposition $A_P^G = A_Q^G \times A_P^Q$

$$A_P^G = \left( \bigcap_{\alpha \in \Delta_P^Q} \text{Ker } \alpha \right) \times \left( \bigcap_{\alpha \in \Delta_P \setminus \Delta_P^Q} \text{Ker } \beta_\alpha \right) \equiv A_Q^G \times A_P^Q$$

Example:

$$\Delta_P^Q = \{ \alpha_1 \}$$
Step 1: Partial Bordification of $A_P^G$ by a Corner

Use $\Delta_P$ as coordinates:

$$A_P^G \sim (\mathbb{R}^+) \Delta_P, \quad a \mapsto (a^{\alpha_1}, \ldots, a^{\alpha_r}).$$

Let each $a^{\alpha_i}$ attain infinity to define the corner:

$$\bar{A}_P^G \sim (\mathbb{R}^+ \cup \{\infty\}) \Delta_P, \quad a \mapsto (a^{\alpha_1}, \ldots, a^{\alpha_r}).$$

Properties of corners:

- Real analytic structure: $\bar{A}_P^G \cong (\mathbb{R}_{\geq 0}) \Delta_P, \quad a \mapsto (a^{-\alpha_1}, \ldots, a^{-\alpha_r}).$

- For $Q \supseteq P$, $\bar{A}_Q^G \hookrightarrow \bar{A}_P^G$.

- Stratified by $A_P^G$-orbits: set $o_Q = \{\infty\}^{\Delta_P} \setminus \Delta_P^Q \times \{1\}^{\Delta_P^Q}$. Then

$$\bar{A}_P^G = \bigsqcup_{Q \supseteq P} A_P^G \cdot o_Q = \bigsqcup_{Q \supseteq P} A_P^Q \cdot o_Q.$$
Example:

\[ o_P = A_P^P \cdot o_P \]

\[ A_{Q1}^Q \cdot o_Q \]

\[ A_{G+}^Q \]

\[ -A_{Q1}^Q \]

\[ A_{G+}^Q \]

\[ A_{Q2}^Q \]

\[ -A_{P}^Q \]

\[ A_{Q2}^Q \cdot o_Q \]
View the Corner as a Cone on $|\Delta_P|$

Let $|\Delta_P| := (r - 1)$-simplex with vertex set $\Delta_P$ and open faces $|\Delta_P^Q|^\circ$.

Stratification of the cone on $|\Delta_P|:

$$ c(|\Delta_P|) = c(\emptyset) \sqcup \bigsqcup_{Q \supsetneq P} c(|\Delta_P^Q|^\circ), $$

where $c(\emptyset) := \text{vertex}$ and $c(|\Delta_P^Q|^\circ) := c(|\Delta_P^Q|^\circ) \setminus c(\emptyset)$.

- There is a homeomorphism of stratified spaces

$$ \bar{A}_P^G \cong c(|\Delta_P|) $$

$$ A_P^Q \cdot o_Q \longleftrightarrow c(|\Delta_P^Q|^\circ) $$

$$ o_P \longleftrightarrow c(\emptyset) $$
Geodesic Action of $A_P$ on $D$

Basepoint $\implies K$, a Cartan involution $\theta$, and $\theta$-stable lift $\tilde{L}_P$.

Since $G = PK$ (Gram-Schmidt), $P$ acts transitively on $D$.

**Geodesic action** of $a \in A_P$ on $x \in D$:

$$a \circ x \equiv p\tilde{a}KA_G,$$

where $x = pKA_G$ and $\tilde{a} \in \tilde{A}_P$ is the lift of $a$.

Properties of geodesic action:

- independent of the choice of basepoint,
- commutes with the usual action of $P$,
- $A_P^G$ acts freely.
$A_P^G := 0P \backslash D$ \hspace{1em} (quotient by usual action),
$e_P := A_P^G \backslash D$ \hspace{1em} (quotient by geodesic action)
$:= P/K_P \tilde{A}_P$.

Isomorphism of $(A_P^G \times 0P)$-homogeneous spaces:

$D \cong A_P^G \times e_P$.

$A_P^G$ is an affine version of $A_P^G$:

$A_P^G \cong A_P^G$ \hspace{1em} given a choice of basepoint.

Set $\tilde{A}_P^G := A_P^G \times A_P^G \tilde{A}_P^G$.

**Example:**

Upper half-plane $= \mathbb{R}^+ \times \mathbb{R}$

$= \{\text{horocycles at } \infty\} \times \{\text{geodesics tending to } \infty\}$
Step 2: Partial Bordification $D(P)$ of $D$

Replace $A^G_P$ in

$$D \cong A^G_P \times e_P$$

by $\tilde{A}^G_P$ to define:

$$D(P) := \tilde{A}^G_P \times e_P.$$

This is a manifold-with-corners.

The stratification of $\tilde{A}^G_P$ induces a stratification of $D(P)$:

$$D(P) = \bigsqcup_{Q \supseteq P} A^Q_P \cdot o_Q \times e_P.$$
Step 3: Combine Partial Bordifications into $\overline{D}$

Observe we can naturally identify

$$A_P^Q \cdot o_Q \times e_P \equiv e_Q$$

so that

$$D(P) = \bigsqcup_{Q \supseteq P} e_Q.$$ 

Thus for $P \subseteq Q$, $D(Q) \subseteq D(P)$.

Define the bordification

$$\overline{D} := \bigcup_{P} D(P) = \bigsqcup_{P} e_P,$$

where $P$ ranges over all parabolic $\mathbb{Q}$-subgroups (including $G$).
The action of $G$ on $D$ extends to an action of $G(\mathbb{Q})$ on $\overline{D} = \bigsqcup P \cdot e_P$.

The **Borel-Serre compactification** is defined by

$$\overline{X} := \Gamma \backslash \overline{D}.$$ 

The stabilizer in $\Gamma$ of a stratum $e_P$ is $\Gamma_P := \Gamma \cap P$. Thus we have a stratification:

$$\overline{X} = \bigsqcup_{P \in \mathcal{P}} Y_P,$$

where $Y_P := \Gamma_P \backslash e_P$. 
Cylindrical Sets and Links of $\overline{X}$

Pick $s_P \in A_P^G$ and $\tilde{O}_P \subseteq e_P$ such that $O_P := \Gamma_P \setminus \tilde{O}_P \subseteq Y_P$ is relatively compact. Set

$$\tilde{W}_P := (\bar{A}_P^{G^+} \cdot s_P) \times \tilde{O}_P \subseteq D(P)$$

Reduction theory $\implies$ for $s_P$ large, the identifications on $\tilde{W}_P$ induced by $\Gamma$ agree with those induced by $\Gamma_P$. Thus

$$W_P := \Gamma_P \setminus \tilde{W}_P = (\bar{A}_P^{G^+} \cdot s_P) \times O_P$$

is a subset of $\overline{X}$, a **cylindrical set**.

Set $O_P = B$. Since $\bar{A}_P^{G^+}$ is homeomorphic to $c(|\Delta_P|)$,

$$W_P \cong B \times c(|\Delta_P|).$$

Thus the link of the stratum $X_P \subseteq \hat{X}$ is $|\Delta_P|$. 

The Reductive Borel-Serre Compactification $\hat{X}$

Recall that $e_P$ is a $0^P$-homogeneous space.

One can extend the geodesic action of $A_P$ to $L_P$; this commutes with the ordinary action of $N_P$ (S., 2001).

$$\mathcal{N}_P := 0^L_P \backslash e_P$$ (quotient by geodesic action),

$$D_P := N_P \backslash e_P$$ (quotient by usual action)

$$= L_P / K_P A_P,$$ symmetric space associated to $L_P$.

Isomorphism of $(0^L_P \times N_P)$-homogeneous spaces:

$$e_P \cong D_P \times N_P, \quad nmK_P \mapsto (mK_P, n), \quad (n \in N_P, \, m \in 0^L_P).$$

**Example:** For $G = \text{GL}_n(\mathbb{R})$, $D_P =$ product of spaces of positive definite $n_i \times n_i$ symmetric matrices modulo homothety.
Take quotient of $e_P \cong D_P \times N_P$ by $\Gamma_P$ in two stages:

(i) Take quotient by $\Gamma_{NP} := \Gamma \cap N_P$:

$$\Gamma_{NP} \backslash e_P = D_P \times N'_P,$$

where $N'_P \equiv \Gamma_{NP} \backslash N_P$.

(ii) $\Gamma_{LP} := \Gamma_P / \Gamma_{NP}$ acts on the first factor while conjugating the second factor. Thus we have a flat $N'_P$-bundle, the \textbf{nilmanifold fibration}:

$$Y_P = \Gamma_P \backslash e_P \longrightarrow X_P := \Gamma_{LP} \backslash D_P$$

The \textbf{reductive Borel-Serre compactification} $\hat{X}$ is obtained by collapsing the fibers of the nilmanifold fibration on each stratum $Y_P$ of $\overline{X}$. Thus there is stratification:

$$\hat{X} = \coprod_{P \in \mathcal{P}} X_P.$$ 

The closure of a stratum $X_P$ is $\hat{X}_P$. 
Neighborhoods and Links of \( \hat{X} \)

Fix \( x \in X_P \subseteq \hat{X} \); over \( x \) is is a full nilmanifold fiber \( \mathcal{N}'_P \).

Cylindrical neighborhood of this fiber is

\[
\tilde{U} = B \times \mathcal{N}'_P \times c(\{|\Delta_P|\}).
\]

Stratification:

\[
\tilde{U} \cap Y_Q = \begin{cases} 
B \times \mathcal{N}'_P \times c(\{|\Delta_P|\}) & Q = G, \\
B \times \mathcal{N}'_P \times c(\{|\Delta_P^Q|\}) & P < Q < G, \\
B \times \mathcal{N}'_P \times c(\emptyset) & Q = P,
\end{cases}
\]

Project \( \tilde{U} \) to \( \hat{X} \) to obtain a neighborhood \( U \) of \( x \):

\[
U \cap X_Q = \begin{cases} 
B \times \mathcal{N}'_P \times c(\{|\Delta_P|\}) & Q = G, \\
B \times \mathcal{N}'_P^Q \times c(\{|\Delta_P^Q|\}) & P < Q < G, \\
B \times c(\emptyset) & Q = P,
\end{cases}
\]

where \( \mathcal{N}'_P^Q \equiv (\Gamma_{NP}/\Gamma_{NQ})\backslash(N_Q\backslash\mathcal{N}_P) \).
Thus the link of the stratum $X_P \subseteq \tilde{X}$ is

$$\left(\mathbb{N}_P' \times |\Delta_P|\right)/\sim = \bigsqcup_{Q>P} \mathbb{N}^{Q}_P \times |\Delta^{Q}_P|^{\circ},$$

where $(\Gamma_{N_P}n, a) \sim (\Gamma_{N_P}n', a)$ if $a \in |\Delta^{Q}_P|^{\circ}$ and $n' = un$ for some $u \in N_Q.$
Satake Compactifications

• Fix an irreducible representation \((\sigma, U)\) of \(G\).

• Construct compactification \(\mathbb{R}D^*\) of \(D\).

• Decompose \(\mathbb{R}D^*\) into real boundary components.

• Assume \((\sigma, U)\) is geometrically rational and set \(D^*\) to the union of the rational boundary components.

• Define the Satake topology on \(D^*\).

• Set \(X^* := \Gamma \backslash D^*\).

There is a simpler description of the topology due to Zucker.

Assume \((\sigma, U)\) is nontrivial on each \(\mathbb{R}\)-simple component.

Give \(U\) an admissible inner product:
\[
\sigma(g)^* = \sigma(\theta g)^{-1}, \quad \text{for all } g \in G.
\]
Set

\[
\operatorname{Herm}(U) := \text{Hermitian endomorphisms of } U.
\]

\(G\) acts on \(\operatorname{Herm}(U)\):
\[
S \mapsto \sigma(g)S\sigma(g)^*, \quad \text{for } S \in \operatorname{Herm}(U), \ g \in G.
\]

Note that \([I] \in \mathbb{P}\operatorname{Herm}(U)\) is fixed by \(KA_G\), so
\[
D = G/KA_G \hookrightarrow \mathbb{P}\operatorname{Herm}(U), \quad x = gKA_G \mapsto g \cdot [I] = \sigma(g)\sigma(g)^*.
\]

The Satake compactification \(\mathbb{R}D^*\) of \(D\) is the closure of the image.
**Important Case:** Hermitian symmetric spaces.

Here $\mathbb{R}$-root system is always type $C_r$ or $BC_r$.

Let highest $\mathbb{R}$-weight of $(\sigma,U)$ be fundamental weight dual to unique simple $\mathbb{R}$-root $\alpha_r$ at the end of the diagram with a double bond, e.g.

\[
\begin{array}{cccccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 & \delta = \text{h.w.} \ V
\end{array}
\]

Then $\mathbb{R}D^*$ is homeomorphic to the closure of $D$ in $\mathbb{C}^N$ under Harish-Chandra’s embedding as a bounded symmetric domain, the **natural compactification**.
Real Boundary Components

$\mathbb{R}D^*$ is a union of real boundary components:

$$D_{P,h} \equiv \text{Fix}(N_P) \subseteq \mathbb{R}D^*, \quad P \text{ a parabolic } \mathbb{R}\text{-subgroup.}$$

$L_P = P/N_P$ acts on $D_{P,h}$; set

$$L_{P,\ell} := \text{the subgroup fixing } D_{P,h} \text{ pointwise,}$$
$$L_{P,h} := L_P/L_{P,\ell},$$
$$\tilde{L}_{P,h} := \text{lift of } L_{P,h} \text{ to } L_P.$$

We have a decomposition

$$L_P = \tilde{L}_{P,h} \times L_{P,\ell},$$

and hence a decomposition of the associated symmetric spaces,

$$D_P = D_{P,h} \times D_{P,\ell}.$$
Example: the natural compactification of a bounded symmetric domain in $\mathbb{C}^N$.

real boundary components $= \text{holomorphic arc components}$

two points are in the same holomorphic arc component if there are a finite number of holomorphic maps of $\{z \in \mathbb{C} \mid |z| < 1\}$ into the closure of $D$ for which the union of the images is connected and contains the two points.

Each real boundary component is again a bounded symmetric domain.
The Simple Roots of $L_P = \tilde{L}_{P,h} \times L_{P,\ell}$

$\mathbb{R}A :=$ maximal $\mathbb{R}$-split torus $\mathbb{R}A$ of $L_P$ lifted to $G$,

$\mathbb{R}\Delta :=$ simple $\mathbb{R}$-roots of $G$,

$\mathbb{R}\mu :=$ highest $\mathbb{R}$-weight of $U$,

$\mathbb{R}\Delta^P :=$ simple $\mathbb{R}$-roots of $L_P$.

Decompose

$$\mathbb{R}\Delta^P = \kappa(\mathbb{R}\Delta^P) \sqcup \zeta(\mathbb{R}\Delta^P),$$

where $\kappa(\mathbb{R}\Delta^P)$ is the largest subset of $\mathbb{R}\Delta^P$ such that $\kappa(\mathbb{R}\Delta^P) \sqcup \{\mathbb{R}\mu\}$ is connected and $\zeta(\mathbb{R}\Delta^P)$ is its complement.

Then $\tilde{L}_{P,h}$ has simple $\mathbb{R}$-roots $\kappa(\mathbb{R}\Delta^P)$ and $L_{P,\ell}$ has simple $\mathbb{R}$-roots $\zeta(\mathbb{R}\Delta^P)$. 
Normalizers of Real Boundary Components

It can happen that $D_{P',h} = D_{P,h}$ for $P' \neq P$

$$P' \supseteq P \text{ and } \kappa(\mathbb{R} \Delta^{P'}) = \kappa(\mathbb{R} \Delta^{P}) \implies D_{P',h} = D_{P,h}.$$ 

Let $P^\dagger$ be the largest such $P'$; it is the normalizer of $D_{P,h}$. Such a parabolic $\mathbb{R}$-subgroup is called saturated.

Set

$$\omega(\mathbb{R} \Delta^{P}) := \mathbb{R} \Delta^{P^\dagger}$$

largest subset $\psi$ such that $\kappa(\psi) = \kappa(\mathbb{R} \Delta^{P})$. 

Example: \( \mathcal{G} = \text{Sp}_{18}(\mathbb{R}) \) and the Natural Compactification

\[
\mathbb{R} \Delta : \quad \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \alpha_8
\]

\[
\mathbb{R} \Delta^P : \quad \alpha_1 \alpha_2 \alpha_4 \quad \alpha_6 \alpha_7 \alpha_8
\]

\[
\zeta(\mathbb{R} \Delta^P) \quad \kappa(\mathbb{R} \Delta^P)
\]

\[
\omega(\mathbb{R} \Delta^P) = \mathbb{R} \Delta^{P^\dagger} : \quad \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_6 \alpha_7 \alpha_8
\]

\[
\zeta(\mathbb{R} \Delta^{P^\dagger}) \quad \kappa(\mathbb{R} \Delta^{P^\dagger})
\]
What are Rational Boundary Components?

What boundary components should be included in $D^*$ so that $\Gamma$ acts discontinuously on $D^*$ and $\Gamma \backslash D^*$ is compact?

- **$D^*$ should be big enough**: Call a real boundary component $D_{P,h}$ **rationally visible** if $D_{P,h} \subseteq \overline{\Omega} \subseteq \mathbb{R}D^*$, the closure of a Siegel fundamental domain $\Omega$. For the quotient $\Gamma \backslash D^*$ to be compact, we wish to include all rationally visible boundary components.

- **$D^*$ should not be too big**: Call a real boundary component **geometrically rational** if its normalizer $R$ is defined over $\mathbb{Q}$ and if $L_{R,\ell}$ is defined over $\mathbb{Q}$ modulo a compact subgroup. In order to ensure that $\Gamma_{L_{R,h}} := \Gamma \cap L_R / \Gamma \cap L_{R,\ell}$ is arithmetic and hence acts discontinuously on $D_{R,h}$, we only want to include geometrically rational boundary components.
One can show:

- Any geometrically rational boundary component is rationally visible.

To proceed we must assume conversely that $\mathbb{R}D^*$ is geometrically rational (Casselman, 1997):

- Any rationally visible boundary component is geometrically rational.

In this case the geometrically rational boundary components will simply be called **rational boundary components**; they are simply those whose normalizer $P^\dagger$ is defined over $\mathbb{Q}$. Define

\[ D^* \equiv \bigcup_{\text{rational boundary components}} D_{R,h}. \]
When Does Geometric Rationality Hold?

Satake compactifications where geometric rationality is known:

- (Borel, 1962) if $V$ is strongly $\mathbb{Q}$-rational,

- (Baily and Borel, 1966) the Baily-Borel-Satake compactification,

- (Casselman, 1997) if a certain criterion based on the Tits index of $G$ is satisfied,

- (S., 2002) any Satake compactification for which all real boundary components are equal-rank: $\text{rank } G_{P,h} = \text{rank } K_{P,h}$ (except for some specified $\mathbb{Q}$-rank 1 and 2 cases),

- (S., 2003) if $V$ is $\mathbb{Q}$-rational.
Let $D^*$ have the **Satake topology** (or see Zucker’s version on next slide). The **Satake compactification** $X^*$ is defined by

$$X^* = \Gamma \backslash D^*;$$

it has a stratification

$$X^* = \bigsqcup_{R \in \mathcal{P}^*} X_{R,h}$$

where

$$\mathcal{P}^* := \text{set of } \Gamma\text{-equivalence classes of saturated parabolic } \mathbb{Q}\text{-subgroups},$$

and

$$X_{R,h} := \Gamma L_{R,h} \backslash D_{R,h}.$$ 

**Example:** $D$ Hermitian symmetric and $\mathbb{R}D^*$ the natural compactification.

Here Baily and Borel (1966) showed that $X^*$ is a normal projective algebraic variety defined over a number field.
Zucker’s Description of the Topology on $X^*$

For all $P \in \mathcal{P}$ there is a projection

$$D_P = D_{P,h} \times D_{P,\ell} \longrightarrow D_{P,h} = D_{P^\dagger,h}.$$  

This induces a map on arithmetic quotients,

$$\pi_P: X_P \longrightarrow X_{P^\dagger,h},$$

a flat bundle with typical fiber $X_{P,\ell} \equiv \Gamma_{L_P,\ell} \backslash D_{P,\ell}$; if we replace $\Gamma_{L_P}$ by a finite index subgroup it becomes trivial.

Combine the various $\pi_P$ and define

$$\pi: \hat{X} \rightarrow X^*;$$

Zucker proves that the topology on $X^*$ is the quotient topology induced by $\pi$. 