COHOMOLOGY OF LOCALLY SYMMETRIC SPACES AND THE MODULI SPACE OF CURVES

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Let $\Gamma$ be a group and $E$ a $\Gamma$-module. We are interested in the cohomology $H^\cdot(\Gamma; E)$. If $X$ is a contractible space on which $\Gamma$ acts properly one may represent this cohomology topologically as $H^\cdot(\Gamma \backslash X; \mathcal{E})$ for a certain sheaf $\mathcal{E}$. Our primary interest is when $\Gamma$ is arithmetic and $X$ is a symmetric space or when $\Gamma$ is a mapping class group and $X$ is Teichmüller space. When $M = \Gamma \backslash X$ is a compact Riemannian manifold it is profitable to represent cohomology by harmonic forms from which one can prove Poincaré duality. When $M$ is non-compact the same can essentially be done but for the $L^2$-cohomology $H^\cdot_{(2)}(M; \mathcal{E})$, which is an invariant of the quasi-isometry class of the metric.

We consider three examples to indicate that $H^\cdot_{(2)}(M; \mathcal{E})$ can represent a topological invariant: (1) Cheeger’s horn metrics on triangulated pseudomanifolds; (2) Saper’s proof that the $L^2$-cohomology of the Weil-Petersson metric on the moduli space of curves $\mathcal{M}_g$ is the cohomology of the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$; and (3) Zucker’s conjecture (proved by Saper-Stern and Looijenga) that the $L^2$-cohomology of a Hermitian locally symmetric space $\Gamma \backslash X$ is the middle perversity intersection cohomology of the Baily-Borel Satake compactification $\Gamma \backslash X^\ast$. We conclude this section with a heuristic for Zucker’s conjecture.

Example (2) above answered a question of Hain and Looijenga, perhaps motivated in analogy with Zucker’s conjecture. A better analogy suggests one consider the Siegel metric on $\mathcal{M}_g$, the pull-back of the locally symmetric metric under the Torelli embedding $\tau: \mathcal{M}_g \to \mathcal{A}_g$, and the Satake compactification $\mathcal{M}^\ast_g$, the closure of $\tau(\mathcal{M}_g)$ in the Baily-Borel Satake compactification $\mathcal{A}^\ast_g$. We conjectured in 1993 that the analogue of Zucker’s conjecture holds in this setting. Although no progress has been made on this conjecture for $g > 3$, more recent work on Rapoport’s conjecture suggests a possible approach.

Rapoport’s conjecture (made independently by Goresky and MacPherson) asserts that for a Hermitian symmetric space $X$, either middle perversity intersection cohomology of the reductive Borel-Serre compactification $\Gamma \backslash X^{\text{RBS}}$ is isomorphic to the middle perversity intersection cohomology of $\Gamma \backslash X^\ast$. The conjecture was motivated by Langlands’s program—the point is that $\Gamma \backslash X^{\text{RBS}}$ is a far less singular compactification making local calculations easier. Saper proved the conjecture (actually a generalization to equal-rank spaces) in 2001 using $L$-modules, a combinatorial model of sheaves on $\Gamma \backslash X^{\text{RBS}}$.

In current work $L$-modules are being used to study $H^\cdot(\Gamma; E)$ itself for $\Gamma$ arithmetic. We now suggest that an analogue of $L$-modules can be applied to address the 1993 conjecture on the moduli space of curves. Namely $\overline{\mathcal{M}_g}$ could play the role of the reductive Borel-Serre compactification. What is needed is to understand the

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Siegel metric locally on $\overline{\mathcal{M}}_g$ (as opposed to locally on $\mathcal{M}_g^*$), understand the fibers of the extended Torelli map $\overline{\mathcal{M}}_g \to \mathcal{M}_g^*$, and prove a vanishing theorem on these fibers. Progress in some simple concrete examples has been made.

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